The Linear-in-Means Model with Heterogeneous Interactions*

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Abstract

We study peer effects in linear-in-means models with heterogeneous interaction effects. The classical linear-in-means model imposes strict homogeneity on the interaction effects, yielding testable implications that can be readily examined in data. We relax these restrictions to allow for both positive and negative interaction effects that can vary within and across groups. This extension makes the linear-in-means model suited to study a wide range of economic behaviors in addition to peer effects, such as joint labor supply decisions within households and strategic interactions among firms. We analyze what can and cannot be learned from frequently used OLS and IV estimands for linear-in-means models once interaction effects are heterogeneous. Although these estimands no longer deliver point identification, we show they can still be used to draw inferences about key economic quantities. We apply these results to data from two economic settings: classroom peer effects in Kenyan primary schools and strategic pricing among cocoa traders in Sierra Leone. In each application, we reject homogeneous interaction effects. Nevertheless, we still draw meaningful inferences about endogenous interactions and social multipliers while allowing for heterogeneous interaction effects.

1 Introduction

Peer effects models are widely used in economics to study how individuals' actions are shaped by those around them, with applications ranging from education and health to labor markets and beyond. The classical linear-in-means model (Manski, 1993) remains the most commonly used framework for empirically analyzing these interactions.[1] This model typically assumes

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[1] The model has been used to study spillover effects in education (Sacerdote, 2001), risk sharing (Townsend, 1994), program participation (Dahl et al., 2014), labor productivity (Guryan et al., 2009), crime (Patacchini & Zenou, 2009), strategic pricing (Casaburi & Reed, 2022), and much more. See Sacerdote (2011), Epple & Romano (2011), Blume et al. (2011), de Paula (2017), and Bramoullé et al. (2020) for surveys of applications.

strict homogeneity in the endogenous interaction effects, requiring that all individuals, within and across peer groups, are influenced in the same way by the average outcome of their peers. Identification and estimation are well-studied under this homogeneity assumption, with researchers relying on linear OLS and IV estimators to recover economic quantities of interest; see Kline & Tamer (2020) for a recent survey. However, the identification arguments behind these estimators do not readily transfer to settings with unobserved heterogeneity in interaction effects, where individuals respond in different ways to the mean behavior of their peers.

The goal of our paper is to study peer effects in linear-in-means models with unobserved heterogeneity in endogenous interaction effects. Our main contribution is to characterize the identifying content of frequently used OLS and IV estimands for linear-in-means models once the standard homogeneity restriction is relaxed. Although these estimands no longer deliver point identification, we show they are still informative about key economic quantities, even when available instruments are binary or have limited support. This stands in contrast with existing approaches to identification under heterogeneous effects, e.g., Masten (2017), which place strong demands on available instruments and can be difficult to implement in practice.

Our analysis focuses on a setting with multiple peer groups, where each group g contains a set of agents $\mathcal{N}_g$. An agent i's action Y_{ig} depends linearly on the average action of her peers:

$$Y_{ig} = \alpha_{ig} + \frac{\beta_{ig}}{|\mathcal{N}_g| - 1} \sum_{j \neq i} Y_{jg} + Z'_{ig}\gamma_{ig}, \quad \text{for } i \in \mathcal{N}_g.$$

In this model, $(\alpha_{ig}, \beta_{ig}, \gamma_{ig})$ are unknown structural parameters, and Z_{ig} is an observed vector, which can include both individual-level shifters and group-level covariates.[2] The parameter β_{ig} is the interaction effect, measuring how agent i in group g responds to the mean outcome of her peers. The classical linear-in-means model assumes β_{ig} is constant among agents, both within and across groups. We instead allow it to vary along both dimensions. We also place no restrictions on the sign or magnitude of β_{ig}, so peer outcomes can affect agents positively or negatively, with arbitrary intensity. The parameters α_{ig} and γ_{ig} specify how Z_{ig} affects Y_{ig} in the absence of spillovers. We allow these objects to vary flexibly within and across groups. Moreover, we do not restrict the size or composition of peer groups, as characterized by $\mathcal{N}_g$.

In Section II, we begin by reviewing the economic quantities commonly studied in models with constant effects, along with the identification strategies used to recover these quantities. To guide and interpret our results, we draw on three examples: classroom peer effects, household labor supply decisions, and competition among firms in oligopolies. In each example, we show that the assumption of constant interaction effects imposes strong restrictions on preferences or technology. Allowing for heterogeneous effects relaxes these restrictions, making the model suited to study a broader set of economic questions. We also show that the constant effects model yields testable implications in the form of overidentifying restrictions and restrictions on OLS estimands, which can be used to assess homogeneity of interaction effects.

[2] An element of Z_{ig} is a shifter if $Z_{ig} \neq Z_{jg}$ for $i \neq j$, and a group-level covariate if $Z_{ig} = Z_{jg}$ for $i, j \in \mathcal{N}_g$.

Motivated by this analysis, we consider, in Section III, the heterogeneous effects model, which allows α_{ig}, β_{ig}, and γ_{ig} to vary freely among agents, within and across groups. Under this more general framework, we derive closed-form expressions for the equilibrium outcomes in terms of individual interaction effects. These expressions allow us to characterize reduced form effects under heterogeneity, uncovering how the signs, magnitudes, and configuration of interaction effects shape the propagation of shocks. We show that, with heterogeneous effects, the equilibrium impact of exogenous shocks on group-level outcomes depends on which agents in the peer group are directly exposed. These equilibrium effects may also vary across groups.

We then examine what features of the model are recovered from OLS and IV estimation under heterogeneous effects. We first study a class of OLS estimands obtained by regressing outcomes Y on exogenous variables Z (or linear combinations of Z). We show that, when specified correctly, these regressions recover averages of equilibrium effects of Z on Y across groups. OLS regressions can also be used to learn about social multiplier effects that tell us how endogenous interactions distort the impact of individual shocks on group-level outcomes (Glaeser et al., 2003). While OLS does not point identify social multipliers under heterogeneous effects, we show it can still be used to assess the signs of these multipliers, allowing us to infer whether spillovers amplify or attenuate the impacts of targeted policies in networks. In addition, we show how OLS can be used to test for positive and negative interaction effects.

We next analyze what features of the model are recovered from IV estimation. We study a large class of IV estimands that use excluded variation to recover the endogenous interaction effects. We show that, under heterogeneity, the IV estimand represents a weighted average of interaction effects, which assigns greater weight to groups in which peer outcomes are more responsive to instruments. We derive necessary and sufficient conditions for the weights to be non-negative, which we view as a minimal requirement for the IV estimand to be informative about interaction effects. Under this condition, we explain how to use the weighted-average interpretation to bound an unweighted average of interaction effects. We show that in many common network models, such as classical peer effects, oligopolies, and public goods games, the IV estimand recovers an upper bound on the magnitude of the average interaction effect.

In Section IV, we apply our analysis to data from two economic studies that employ the linear-in-means model with homogeneous effects: peer effects within Kenyan primary schools (Duflo et al., 2011) and competition among cocoa traders in Sierra Leone (Casaburi & Reed, 2022). In both settings, we find evidence to reject homogeneous interaction effects. In the first application, we find that peer effects differ within and across classrooms. In the second application, we find that traders respond strategically in different ways to their rivals' actions.

Given our findings, we then reanalyze these two empirical applications under the linear-in-means model with heterogeneous interaction effects. In the Kenyan primary school setting, we find evidence of positive peer effects among a large share of students. Our estimate of the upper bound on the average peer effect indicates that a one-point increase in peers' average test scores raises a student's own test score by no more than 0.45 points, on average. We also find strong evidence of social multiplier effects, implying that the peer interactions amplify the impacts of policies targeting individual student achievement. In the analysis of strategic pricing in Sierra Leone, we strong find evidence of strategic interactions. Our estimated upper bound on the average conduct parameter implies that raising rivals' cocoa purchases by one pound reduces a trader's own purchases by less than 0.02 pounds, on average. We find no evidence of social multiplier effects in this setting, suggesting that strategic interactions do not substantially alter how trader-specific changes in demand or costs affect total market output.

Our paper contributes to two literatures. First, we contribute to a literature on the econometric analysis of linear peer effects models; see de Paula (2017), Kline & Tamer (2020), and Bramoullé et al. (2020) for recent surveys. In this literature, there is growing recognition of the importance of accounting for heterogeneity in endogenous interaction effects.[3] We study a setting with unobserved heterogeneity, allowing interaction effects to vary in both sign and magnitude, within and across groups. This setup differs from network-based approaches to heterogeneous peer exposure (Bramoullé et al., 2009; Blume et al., 2015), where heterogeneity is encoded in an adjacency matrix that is either known to the researcher or identifiable from repeated observations of the same network (de Paula et al., 2025). In those settings, the heterogeneous interaction structure is observed or recoverable, and the main object is still a common peer effect parameter conditional on that structure.[4] Our paper takes a different approach, treating the interaction effect itself as an unobserved random variable, and studying what researchers can learn about its distribution from data on heterogeneous peer groups.

In this regard, our paper speaks to Masten (2017), who studies identification of a linear peer effects model with random coefficients.[5] He proves that the marginal distributions of the coefficients are point identified if there is an instrument with continuous variation over a large support. However, he also shows that instruments are insufficient for recovering the joint distribution of random coefficients. This limitation is important because many economic quantities of interest, including equilibrium effects and social multipliers, depend on joint features of these coefficients rather than on their marginals alone. It therefore raises questions about what can be learned about these economic quantities under heterogeneity. Our paper addresses this question by analyzing how to interpret and learn from frequently used OLS and IV estimators in the presence of heterogeneous interaction effects. We view our results as constructive: although point identification is generally unavailable, these estimators still support meaningful inference. Our tools can be applied in many common empirical settings,

[3] Both Sacerdote (2011) and Bramoullé et al. (2020) emphasize the relevance of heterogeneity in interaction effects. Volpe (2025) shows empirical evidence on heterogeneous interaction effects in discrete choice models.

[4] Bramoullé et al. (2009) and Blume et al. (2015) study linear peer effects models where known restrictions on network links identify a common positive peer effect. de Paula et al. (2025) derive conditions for the links themselves to be identified from panel data with repeated observations of the same network. Neither of these strategies apply to our setting where interaction effects are unobserved and heterogeneous in the population.

[5] Hurwicz (1950) also studies simultaneous equation models with random coefficients, though he does not provide explicit identification results. Kelejian (1974) and Hahn (2001) also analyze related models, though Masten (2017) points out that the analysis in both of these papers is based on self-contradictory assumptions.

including those where access to a continuous instrument with large support is not feasible.

The second literature we contribute to is concerned with the interpretation of linear OLS and IV estimands in settings with unobserved treatment effect heterogeneity; see Mogstad & Torgovitsky (2024) for a recent survey. In a seminal paper, Imbens & Angrist (1994) develop a framework for interpreting IV estimands as weighted averages of local average treatment effects, and Angrist et al. (2000) extend these interpretations to supply and demand models consisting of two simultaneous equations. The system of linear simultaneous equations for peer effects differs in two important ways from the linear supply and demand system studied by Angrist et al. (2000). First, the supply and demand system is restricted to a network of two agents (a representative producer and representative consumer in each market), whereas we study networks of arbitrary size. Second, the supply and demand system focuses on a case where the sign of interaction effects is known (upward-sloping supply and downward-sloping demand). By contrast, the system we study does not restrict the signs of interaction effects: agents' outcomes can be strategic complements or substitutes, allowing the model to accommodate settings such as peer effects, household labor supply, and oligopolistic competition.

Our paper contributes to this literature by demonstrating how to interpret linear OLS and IV estimands for linear peer effects models under heterogeneous interaction effects. Our analysis shows that many of the existing tools for interpreting these estimands do not readily transfer to peer effects models. For example, with peer groups of more than two agents, the standard monotonicity conditions for IV to have a causal interpretation (Imbens & Angrist, 1994) place strong restrictions on the endogenous interactions, which are unlikely to apply in empirical settings. We derive alternative, weaker conditions under which IV retains a causal interpretation. We then use this interpretation to derive bounds on key economic quantities. Overall, our analysis provides a tractable approach to learn about heterogeneous interaction effects and social multipliers from estimands commonly used in empirical peer effects settings.

2 The Linear-in-Means Model

In this section, we introduce the linear-in-means model, provide economic interpretations of the model, and define economic quantities that serve as target parameters in our econometric analysis. We then examine how each of these parameters can be recovered from the data under the assumption that the endogenous interaction effects are homogeneous across individuals.

2.1 Econometric Model

Consider a generalized linear-in-means model that allows for heterogeneous interaction effects:

$$Y_{ig} = \alpha_{ig} + \frac{\beta_{ig}}{|\mathcal{N}_g| - 1} \sum_{j \neq i} Y_{jg} + Z_{ig}' \gamma_{ig}, \quad \text{for } i \in \mathcal{N}_g. \tag{1}$$

Stacking the outcomes, intercepts, and direct effects as Y_g, α_g, and $D_g = (Z_{ig}'\gamma_{ig})_{i \in \mathcal{N}_g}$, we can re-write these equations in matrix form as $Y_g = \alpha_g + \mathbf{B}_g Y_g + D_g$, where $\mathbf{B}_g$ is a matrix given by:

$$\mathbf{B}_g = \frac{1}{|\mathcal{N}_g| - 1} \begin{bmatrix} 0 & \beta_{1g} & \cdots & \beta_{1g} \\ \beta_{2g} & 0 & \cdots & \beta_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{|\mathcal{N}_g|g} & \beta_{|\mathcal{N}_g|g} & \cdots & 0 \end{bmatrix}.$$

We take specification (1) as the estimating equation in our analysis. To understand the range of economic behaviors it allows, we first compare it with the classical linear-in-means model.

Classical Linear-in-Means Assumptions. The classical linear-in-means model imposes four economic assumptions. First, it imposes within-group homogeneity in interaction effects, $\beta_{ig} = \beta_{jg}$ for $i, j \in \mathcal{N}_g$, so all members of each group respond in the same way to the average outcome of their peers. Second, it imposes between-group homogeneity in interaction effects, $\beta_{ig} = \beta_i$ for all i and g, so the interaction effects are identical across groups. Third, it imposes $|\beta_{ig}| < 1$ for all i and g, so the interaction effects are all bounded in magnitude. Fourth, it imposes homogeneous direct effects, $\gamma_{ig} = \gamma$ for all i and g, so the effect of Z_{ig} on Y_{ig} absent spillovers is fixed in the population. While variants of the linear-in-means model differ in the exact assumptions they impose, these four restrictions form the benchmark for our analysis.

Assumptions A (Classical Linear-in-Means Assumptions)**.**

- **A.1** *(Homogeneous Interactions within Groups)***.** $\beta_{ig} = \beta_{jg}$ for any two agents $i, j \in \mathcal{N}_g$.
- **A.2** *(Homogeneous Interactions across Groups)***.** $\beta_{ig} = \beta_i$ for all agents i and groups g.
- **A.3** *(Bounded Interaction Effects)***.** $|\beta_{ig}| < 1$ for all agents i and groups g.
- **A.4** *(Homogeneous Incidence of Z)***.** $\gamma_{ig} = \gamma$ for all agents i and groups g.

Among these assumptions, A.1 and A.2 are most important for us going forward. Together, they rule out heterogeneity in both the sign and magnitude of interaction effects, eliminating settings in which some individuals conform while others differentiate, or in which the strength of peer influence differs among individuals. As we show later, these homogeneity restrictions are central to the conventional identification arguments for the linear-in-means model, which leverage exclusion restrictions within a linear IV framework to recover endogenous interaction effects. Relaxing these restrictions therefore requires an alternative approach to identification.

2.2 Economic Interpretations of the Model

We now illustrate how the linear-in-means model may be derived as the estimating equation for three economic decision problems: peer effects in schools, joint labor supply decisions in households, and strategic interactions among firms in oligopolies. In each example, strong restrictions on preferences or technology are needed to rationalize the classical linear-in-means assumptions. Relaxing these assumptions therefore broadens the model's economic scope. We continue to draw on these examples throughout the paper to guide and interpret our analysis.

2.2.1 Peer Effects

Consider a peer group g in which each individual i chooses an action $Y_{ig} \in \mathbb{R}$. Utility reflects an agent's preference for conforming to, or deviating from, the average behavior of her peers:

$$U_{ig}(Y_{ig}|Z_{ig}, \bar{Y}_{-ig}) = \big(\alpha_{ig} + Z_{ig}'\gamma_{ig}\big)Y_{ig} - \frac{\beta_{ig}}{2}(Y_{ig} - \bar{Y}_{-ig})^2 - \frac{1-\beta_{ig}}{2}Y_{ig}^2.$$

This type of utility function is widely used to study peer effects; see Blume et al. (2015) for discussion. The first term captures the private determinants of the agent's choice, which may depend on both observed and unobserved factors, Z_{ig} and α_{ig}, respectively. The second term is the social payoff, penalizing deviations between one's action and the mean action of one's peers. The third term is a convex cost of action. In equilibrium, agents' choices Y_{ig} satisfy (1).[6]

Under this first interpretation of the model, the interaction effect β_{ig} measures how agent i in group g conforms to or differentiates from her peers. By imposing a common β_{ig}, the classical linear-in-means model assumes that all agents, within and across groups, face the same social incentives, and thus have the same marginal rate of substitution between private and social utility. By relaxing this restriction, our framework accommodates richer patterns of peer influence. For example, some agents may prefer to conform while others may prefer to differentiate; alternatively, all agents may prefer to conform, but with different intensities.

2.2.2 Household Labor Supply

Consider a non-unitary model of household labor supply, as reviewed in the survey by Donni & Chiappori (2011). Members of each household g allocate a fixed time endowment T between work and leisure. Let h_{ig} denote the hours worked by member i, and let W_{ig} denote her wage. Individual labor income is then $Y_{ig} = W_{ig}h_{ig}$. The incomes of household members are pooled and reallocated so that each member i receives some fraction $\kappa_{ig} \in [0, 1]$ to spend on personal consumption. The total value of household consumption, denoted by C_g, cannot exceed total household income. In addition to consuming $\kappa_{ig}C_g$, each individual i can also consume non-transferable goods. These goods may come, for example, in the form of workplace amenities or health benefits that only i can access. Let a_{ig} be the value of these goods to individual i.

Each individual derives utility from consumption and leisure, with diminishing marginal utility in both inputs. We represent preferences by the following log-additive utility function:

$$\max_{h_{ig}} \; U_{ig}(h_{ig}|W_{ig}, C_g) = \mu_{ig}\log(T - h_{ig}) + (1-\mu_{ig})\log(a_{ig} + \kappa_{ig}C_g), \quad \text{s.t.} \quad C_g = \sum_{j\in\mathcal{N}_g} W_{jg}h_{jg}.$$

The parameter $\mu_{ig} \in [0, 1]$ determines an individual i's marginal rate of substitution between consumption and leisure. As long as everyone spends time working, $h_{ig} \in (0, T)$, an interior

[6] An alternate utility specification, adopted by Epple & Romano (1998) and Calvó-Armengol et al. (2009), is $U_{ig}(Y_{ig}|Z_{ig}, \bar{Y}_{-ig}) = \big(\alpha_{ig} + Z_{ig}'\gamma_{ig}\big)Y_{ig} + \beta_{ig}\bar{Y}_{-ig}Y_i - \frac{1}{2}Y_{ig}^2$, which also rationalizes the linear-in-means model.

solution exists. Equilibrium outcomes are characterized by the linear-in-means model in (1):

$$\begin{aligned} Y_{ig} &= -\frac{\mu_{ig} a_{ig}}{\kappa_{ig}} - \mu_{ig} \sum_{j \neq i} Y_{jg} + (1-\mu_{ig}) T W_{ig} \\ &= \underbrace{\alpha_{ig}}_{-\frac{\mu_{ig} a_{ig}}{\kappa_{ig}}} + \underbrace{\frac{\beta_{ig}}{|\mathcal{N}_g| - 1}}_{-\mu_{ig}} \sum_{j \neq i} Y_{jg} + \underbrace{\gamma_{ig}}_{(1-\mu_{ig})T} W_{ig}, \quad \text{for } i \in \mathcal{N}_g. \end{aligned}$$

Under this second interpretation of the model, the interaction effect β_{ig} is an added earner effect, measuring how agent i's income responds to the incomes of other household members. The same parameter also governs the elasticity of i's earnings with respect to her own wage. By imposing a common β_{ig}, the classical linear-in-means model restricts these labor supply responses to be identical within and across households, so each agent's labor supply responds in the same way to the incomes of other household members, as well as to her own wage. By relaxing this restriction, we allow these responses to vary among individuals, for example, between primary and secondary earners or across households with different numbers of children.

2.2.3 Firm Oligopoly

Consider an oligopoly model where firms have heterogeneous, convex cost curves. Following Bresnahan (1981) and Perry (1982), we adopt a conjectural-variations framework that nests both Bertrand and Cournot competition. Under this framework, firms form conjectures about their rivals' best responses, which must then be consistent with their outcomes in equilibrium.

Each market g contains multiple firms i, each producing output Y_{ig}. The price that clears the market is defined by an inverse demand function $P_g = a_g - b_g \sum_{i \in \mathcal{N}_g} Y_{ig}$, where a_g and b_g can vary across markets g. A firm's production costs are $c_{ig}(Y_{ig}) = (\lambda_{ig0} + Z'_{ig}\lambda_{ig1}) Y_{ig} + \frac{1}{2}\delta_{ig} Y_{ig}^2$, where λ_{ig0}, λ_{ig1}, and δ_{ig} can vary both across firms i and across markets g. The vector Z_{ig} contains a set of observable cost-shifters, which can directly impact the firm's productivity.

We suppose that every firm i has some reference output Y_{ig}^0, which is common knowledge in the market. Each firm conjectures that increasing its own output Y_{ig} relative to Y_{ig}^0 causes other firms to adjust their total output by θ_{ig}, anticipating that $\sum_{j \neq i} Y_{jg}$ equals $\sum_{j \neq i} Y_{jg}^0 + \theta_{ig}(Y_{ig} - Y_{ig}^0)$. Given these conjectures, each firm i in market g maximizes its profit by solving:

$$\max_{Y_{ig}} \; \Pi_{ig}(Y_{ig} | Z_{ig}, \{Y_{jg}^0\}_{j \neq i}) = P_g Y_{ig} - c_{ig}(Y_{ig}), \quad \text{s.t.} \quad \begin{aligned} P_g &= a_g - b_g \sum_{i \in \mathcal{N}_g} Y_{ig} \\ \sum_{j \neq i} Y_{jg} &= \sum_{j \neq i} Y_{jg}^0 + \theta_{ig}(Y_{ig} - Y_{ig}^0) \\ c_{ig}(Y_{ig}) &= (\lambda_{ig0} + Z'_{ig}\lambda_{ig1}) Y_{ig} + \frac{1}{2}\delta_{ig} Y_{ig}^2. \end{aligned}$$

In this model, θ_{ig} is a conjectural-variation parameter, measuring firm i's perceived influence on its rivals' output. Three special cases are most notable. First, if $\theta_{ig} = 0$ for all i, the model reduces to Cournot competition, where firms do not internalize their rivals' output decisions. Second, if $\theta_{ig} = -1$ for all i, the model reduces to Bertrand competition, where firms perceive their actions to have no impact on total market output. Third, if $\theta_{ig} = |\mathcal{N}_g| - 1$ for all i, firms act as monopolists. Given this range of cases, we let θ_{ig} take values between -1 and $|\mathcal{N}_g| - 1$.

In equilibrium, each firm's output must equal its reference output, $Y_{ig} = Y^0_{ig}$. The resulting equilibrium conditions generate the linear-in-means model in (1) as an estimating equation:

$$\begin{aligned} Y_{ig} &= \frac{1}{\delta_{ig} + b_g(2+\theta_{ig})}\left[a_g - \lambda_{ig0} - b_g \sum_{j\neq i} Y_{jg} - Z'_{ig}\lambda_{ig1}\right] \\ &= \underbrace{\alpha_{ig}}_{\frac{a_g - \lambda_{ig0}}{\delta_{ig}+b_g(2+\theta_{ig})}} + \underbrace{\frac{\beta_{ig}}{|\mathcal{N}_g| - 1}}_{-\frac{b_g}{\delta_{ig}+b_g(2+\theta_{ig})}} \sum_{j\neq i} Y_{jg} + Z'_{ig} \underbrace{\gamma_{ig}}_{-\frac{\lambda_{ig1}}{\delta_{ig}+b_g(2+\theta_{ig})}}, \quad \text{for } i \in \mathcal{N}_g. \end{aligned}$$

Under this third interpretation of the model, the interaction effect β_{ig} is a conduct parameter, in the sense of Weyl & Fabinger (2013). It measures how each firm i's own output responds to the output of its competitors. This quantity is governed by three primitives: the demand slope b_g, the slope δ_{ig} of firm i's marginal cost curve, and the conjectural-variation parameter θ_{ig}. By imposing a common β_{ig}, the classical linear-in-means model assumes that: (1) consumer demand is equally elastic in all markets, (2) all firms' marginal costs have the same curvature, and (3) all firms exhibit the same beliefs about competition. By allowing for heterogeneous interaction effects, we relax each of these restrictions, permitting firms to differ in technology, in the slopes of their consumer demand curves, and in their beliefs about rivals' behavior.

2.3 Economic Quantities of Interest

Depending on the empirical context, researchers may be interested in learning about a variety of reduced form and structural parameters in the model. In Table 1, we list several economic quantities that are commonly studied in the classical linear-in-means model. For each one, we give a definition and derive its expression in terms of the structural parameters. To ease notation, we suppress group subscripts and set $\mathcal{N} = \{1, \dots, N\}$, while noting that the group size can freely vary in the population. For exposition, we also take Z_i to be a scalar, although allowing for a vector of shifters and/or covariates does not meaningfully impact our analysis.

We begin by analyzing economic quantities under the classical linear-in-means model (in Column 2 of Table 1), deferring their analysis under heterogeneous effects (in Column 3 of Table 1) to Section 3. To analyze these quantities, we first solve for an equilibrium. Under Assumptions A.1-A.4, equations (1) admit a unique solution, which yields the reduced form:

$$Y_i = (1 + \beta\psi)(\alpha_i + \gamma Z_i) + \psi \sum_{j\neq i}(\alpha_j + \gamma Z_j), \quad \text{for } i \in \{1, \dots, N\}. \tag{2}$$

These equations characterize how Z affects Y in equilibrium, after accounting for spillovers.

The parameter ψ is the equilibrium effect of j's outcome Y_j on i's outcome Y_i, and is given by:

$$\psi = \frac{\beta}{(1-\beta)(N-1+\beta)}.$$

Under Assumption A.3, ψ inherits the same sign as β, and it tends to zero as N grows large.

Spillover Effect. The first economic quantity we analyze is the individual spillover effect of Z_j on Y_i. This quantity measures how a change in the shifter faced by agent j affects the equilibrium outcome of agent i. In a peer effects model, it captures how student i's achievement responds indirectly to factors that shift student j's achievement. In a household labor-supply model, it captures how member i's income responds to another household member j's wage. In an oligopoly model, it captures how firm i's output responds to a cost shock at rival firm j.

The spillover effect can be decomposed into the product of two terms, γ and ψ, where γ is the direct effect of Z_j on Y_j in the absence of spillovers, and ψ is the spillover effect of Y_j on Y_i:

$$\frac{\Delta Y_i}{\Delta Z_j} = \gamma \times \psi. \tag{3}$$

Under Assumptions A.1-A.4, $\Delta Y_i/\Delta Z_j$ is constant across all agent pairs (i,j). Therefore, the classical linear-in-means model assumes that the spillover effect of Z_j on Y_i is homogeneous: it does not depend on which individual receives the shock or who is indirectly affected by it.

Total Individual Effect. The second quantity that we study is the total individual effect of Z_i on Y_i. This quantity measures how agent i's shifter affects her own outcome in equilibrium, after accounting for spillovers. In a peer effects model, it captures how student i's achievement responds in equilibrium to an intervention that shifts her own ability. In a household labor-supply model, it captures how agent i's income responds in equilibrium to her own wage. In an oligopoly model, it captures how firm i's output responds in equilibrium to a cost shock.

The total effect of Z_i on Y_i decomposes into a direct effect and an indirect feedback effect:

$$\frac{\Delta Y_i}{\Delta Z_i} = \gamma + \underbrace{\beta \frac{\Delta \bar{Y}_{-i}}{\Delta Z_i}}_{\text{Indirect Effect}}, \quad \text{where} \quad \frac{\Delta \bar{Y}_{-i}}{\Delta Z_i} = \gamma \times \psi. \tag{4}$$

The indirect effect accounts for network distortions, capturing how agent i's behavior reflects back onto itself through interactions with others. The classical linear-in-means model imposes a sharp sign restriction on this feedback term. Specifically, under Assumptions A.1-A.4, the model implies that $\beta\psi \geq 0$, so the indirect effect, $\beta \times (\Delta \bar{Y}_{-i}/\Delta Z_i)$, weakly inherits the same sign as the direct effect, γ. Spillovers therefore always weakly amplify the effect of Z_i on Y_i: $|\Delta Y_i/\Delta Z_i| = |\gamma(1+\beta\psi)| \geq |\gamma|$. As we show later, this property need not hold in more general settings. Under heterogeneous interaction effects, equilibrium feedback can either amplify or attenuate the effect of Z_i on Y_i, depending on the signs and magnitudes of interaction effects.

In addition to restricting the sign of equilibrium feedback, the classical linear-in-means model maintains that the total individual effect is homogeneous in the population. In particular, the model assumes that $\Delta Y_i/\Delta Z_i$ is constant within and across groups, so the equilibrium effect of a change in one's own shifter does not depend on the identity of the affected agent.

Total Effect on the Average. The third quantity we consider is the total effect of Z_i on $\bar{Y}$. This quantity measures how agent i's shifter impacts the average equilibrium outcome in a group, after accounting for spillovers. In a peer effects model, it captures how an intervention targeting student i affects average achievement in a classroom. In a household labor-supply model, it captures how member i's wage affects average household earnings. In an oligopoly model, it captures how a productivity shock to firm i affects average output in the market.

The total effect on the average is an average of total individual effects and spillover effects:

$$\frac{\Delta \bar{Y}}{\Delta Z_i} = \frac{1}{N}\left[\frac{\Delta Y_i}{\Delta Z_i} + \sum_{j \neq i} \frac{\Delta Y_j}{\Delta Z_i}\right]. \tag{5}$$

Under Assumptions A.1-A.4, this quantity reduces to a constant, $\Delta \bar{Y}/\Delta Z_i = \frac{1}{N}\left(\frac{\gamma}{1-\beta}\right)$. Hence, the classical model assumes that the effect of an individual-level shock on the group average does not depend on the identity of the shocked agent. Moreover, since $|\beta| < 1$, any shock that directly raises i's outcome ($\gamma > 0$) must also raise the group average outcome ($\Delta \bar{Y}/\Delta Z_i > 0$).

Social Multiplier Effect. The fourth quantity we define is the social multiplier effect. This quantity measures how endogenous interactions distort the impact of individual-level shocks on group-level outcomes. In the classical linear-in-means model, Glaeser et al. (2003) define the multiplier as a ratio of aggregate coefficients to individual coefficients in the reduced form, thus comparing how the equilibrium impact of Z on Y varies at different levels of aggregation:

$$M = \frac{\Delta \bar{Y}/\Delta \bar{Z}}{\Delta Y_i/\Delta Z_i} = \frac{\beta + N - 1}{\beta + (1-\beta)(N-1)}, \tag{6}$$

As the size of the group N grows large, the social multiplier effect M converges to $(1-\beta)^{-1}$.

Much of the literature on social multipliers assumes positive interaction effects, $\beta \geq 0$; see, for example, Goldin & Katz (2002), Glaeser et al. (2003), and Becker & Murphy (2003). Under this restriction, the multiplier effect is weakly greater than one, $M \geq 1$, so endogenous interactions amplify the impact of individual-level shocks on group-level outcomes. By contrast, when $\beta < 0$, spillovers suppress the impact of individual-level shocks at the group-level, yielding $M < 1$. Once we extend the model to allow for heterogeneous interactions, the social multiplier will no longer be governed by a single interaction parameter; it will instead depend on the full configuration of interaction effects (both their signs and magnitudes) across agents.

Structural Coefficients. The final set of quantities we examine are structural coefficients α_i, β_i, and γ_i. Among these, the interaction effect β_i is typically the primary target parameter for researchers. In a peer effects model, it measures the amount of social pressure agent i faces. In a household labor-supply model, it measures how member i's income responds to the incomes of other household members. In an oligopoly model, it measures the extent of strategic interaction among firms. The other parameters α_i and γ_i describe how Z_i affects Y_i in the absence of spillovers. Such objects are useful in settings where one wants to distinguish direct responses to a shock from the indirect effects transmitted through social interactions.

By imposing common β_i and γ_i, the classical linear-in-means model collapses the set of target parameters available to researchers, precluding analysis of quantities such as $\text{corr}(\beta_i, \beta_j)$ or $\text{corr}(\gamma_i, \gamma_j)$ that describe how peer effects and direct effects vary and co-vary across agents. These objects are relevant for questions such as whether members of the same peer group face similar social pressures, whether household members share similar consumption-leisure tradeoffs, or whether firms in the same market share similar beliefs about competition. Such questions are vacuous in the classical model, but arise naturally under heterogeneous effects.

Table 1: Economic Quantities under the Linear-in-Means Model

	Structural Interpretation	
	Constant Effects	Heterogeneous Effects
Panel A. Reduced Form Quantities		
Spillover Effect ($\Delta Y_i/\Delta Z_j$)	$\frac{\beta\gamma}{(1-\beta)(N-1+\beta)}$	$\frac{\beta_i\gamma_j\nu_{ij}}{(N-1)\det(I-\mathbf{B})}$
Total Individual Effect ($\Delta Y_i/\Delta Z_i$)	$\gamma + \frac{\beta^2\gamma}{(1-\beta)(N-1+\beta)}$	$\gamma_i + \frac{\beta_i\gamma_i\left(\frac{1}{N-1}\sum_{j\neq i}\beta_j\nu_{ij}\right)}{(N-1)\det(I-\mathbf{B})}$
Total Effect on the Average ($\Delta\bar{Y}/\Delta Z_i$)	$\frac{1}{N}\times\frac{\gamma}{(1-\beta)}$	$\frac{1}{N}\times\frac{\gamma_i\nu_i}{\det(I-\mathbf{B})}$
Individual Social Multiplier $\left(\frac{\sum_{j=1}^N \Delta Y_j/\Delta Z_i}{\Delta Y_i/\Delta Z_i}\right)$	$\frac{\beta+N-1}{\beta+(1-\beta)(N-1)}$	$\frac{\nu_i}{\nu_i-\frac{1}{N-1}\sum_{j\neq i}\beta_j\nu_{ij}}$
Aggregate Social Multiplier $\left(\frac{\sum_{i=1}^N \Delta\bar{Y}/\Delta Z_i}{\frac{1}{N}\sum_{j=1}^N \Delta Y_j/\Delta Z_j}\right)$	$\frac{\beta+N-1}{\beta+(1-\beta)(N-1)}$	$\frac{\frac{1}{N}\sum_{i=1}^N\nu_i\gamma_i}{\frac{1}{N}\sum_{j=1}^N\left(\nu_j-\frac{1}{N-1}\sum_{k\neq j}\beta_k\nu_{jk}\right)\gamma_j}$
Panel B. Structural Quantities		
No Interference Outcome ($Y_i\vert(\bar{Y}_{-i}, Z_i)=\mathbf{0}$)	α_i	α_i
No Interference Effect ($\Delta Y_i/\Delta Z_i\vert\bar{Y}_{-i}$)	γ	γ_i
Interaction Effect ($\Delta Y_i/\Delta\bar{Y}_{-i}$)	β	β_i
Interaction Effect Correlation $\left(\text{corr}\left(\frac{\Delta Y_i}{\Delta\bar{Y}_{-i}}, \frac{\Delta Y_j}{\Delta\bar{Y}_{-j}}\right)\right)$	0	$\text{corr}(\beta_i, \beta_j)$

Notes. The reduced form effects $\Delta Y_i/\Delta Z_j$, $\Delta Y_i/\Delta Z_i$, and $\Delta\bar{Y}/\Delta Z_i$ are defined holding Z_j fixed, for $j \neq i$. To ease notation in the last column, we let $\nu_i = \prod_{\ell\neq i}\left(1+\frac{\beta_\ell}{N-1}\right)$ and $\nu_{ij} = \prod_{\ell\notin\{i,j\}}\left(1+\frac{\beta_\ell}{N-1}\right)$ for any i and j.

2.4 Estimands for Constant Effects Models

We now review the OLS and linear IV estimands frequently used to recover economic quantities in the classical linear-in-means model. Table 2 defines each estimand and characterizes its identifying content, both in the classical model and in the heterogeneous effects extension.

We begin by analyzing estimands under the classical linear-in-means model, and defer the analysis under heterogeneous effects to Section 3. Since peer outcomes are jointly determined, identification relies on exclusion restrictions: sources of variation that directly impact some members of a group but not others. These exclusions can arise from policies or cost shifters that directly affect only a subset of agents' outcomes. They can also arise from restrictions on the group structure, whereby certain agents are known to not interact directly with one another. This latter source of excluded variation is studied by Bramoullé et al. (2009). The corresponding identification arguments are shown to hold generically by Blume et al. (2015).

In our setup, policy and cost exclusions can be represented directly as individual-level shifters Z. Restrictions on group structure play a complementary role: together with observables, they determine which functions of Z are excluded from an agent's outcome equation and serve as instruments for peer outcomes. We formalize this construction in Appendix A.3.

As in standard IV analyses of linear-in-means models, we assume α is mean independent of the shifters Z and group composition $\mathcal{N}$, so $\mathrm{E}(\alpha|Z,\mathcal{N})=\mathrm{E}(\alpha)$. We further require that $\mathrm{E}[ZZ']$ is nonsingular and that $\gamma \neq 0$. Mean independence provides instrument exogeneity, ensuring that Z is a valid instrument, while the latter conditions provide rank and relevance.

2.4.1 Frequently Used OLS Estimands

We first examine OLS estimands obtained by projecting individual and group outcomes on individual-level shifters. For any vector X and any scalar outcome Q, we define the estimand:

$$\beta^{\mathrm{OLS}}(Q;X) = \mathrm{E}(XX')^{-1}\,\mathrm{E}(XQ). \tag{7}$$

We consider three OLS estimands, $\beta^{\mathrm{OLS}}(Y_i;\tilde{Z})$, $\beta^{\mathrm{OLS}}(\bar{Y};\tilde{Z})$, and $\beta^{\mathrm{OLS}}(\bar{Y}_{-i};\tilde{Z})$, corresponding to linear regressions of Y_i, $\bar{Y}$, and $\bar{Y}_{-i}$, respectively, on $\tilde{Z}=(1,Z')'$. Under Assumptions A.1-A.4, the coefficient on Z_j in $\beta^{\mathrm{OLS}}(Y_i;\tilde{Z})$ identifies the spillover effect $\Delta Y_i/\Delta Z_j$, while the coefficient on Z_i identifies the total individual effect $\Delta Y_i/\Delta Z_i$. Likewise, the coefficients on Z_i in $\beta^{\mathrm{OLS}}(\bar{Y};\tilde{Z})$ and $\beta^{\mathrm{OLS}}(\bar{Y}_{-i};\tilde{Z})$ identify $\Delta\bar{Y}/\Delta Z_i$ and $\Delta\bar{Y}_{-i}/\Delta Z_i$, respectively. The social multiplier effect is then identified from a ratio of OLS coefficients: $M=N\beta^{\mathrm{OLS}}_{Z_i}(\bar{Y};\tilde{Z})/\beta^{\mathrm{OLS}}_{Z_i}(Y_i;\tilde{Z})$.

A key restriction of the classical linear-in-means model is that the reduced form effects $\Delta Y_i/\Delta Z_j$, $\Delta Y_i/\Delta Z_i$, and $\Delta\bar{Y}/\Delta Z_i$ are constant across agents i and j. In particular, $\Delta Y_i/\Delta Z_j$ is fixed across agent pairs (i,j) with $i \neq j$, while $\Delta Y_i/\Delta Z_i$ and $\Delta\bar{Y}/\Delta Z_i$ are fixed across agents i. This restriction yields testable implications of the model that can be assessed using OLS estimands. In Lemma 1 below, we list two such tests, both straightforward to implement.

Lemma 1. Suppose the linear-in-means model in (1) has a well-defined reduced form. Then:

(i) If Assumptions A.1 holds, then $\beta^{\mathrm{OLS}}_{Z_i}(Y_j;(1,Z')') = \beta^{\mathrm{OLS}}_{Z_i}(Y_k;(1,Z')')$ for all $i \neq j \neq k$.

(ii) If Assumptions A.1 and A.4 hold, then $\beta^{\mathrm{OLS}}_{\bar{Z}_{-i}}(Y_i;(1,Z_i,\bar{Z}_{-i})') = \beta^{\mathrm{OLS}}_{Z_i}(\bar{Y}_{-i};(1,Z_i,\bar{Z}_{-i})')$.

Lemma 1, Part (i), provides a direct test of Assumption A.1, which maintains that interaction effects are constant among agents within a group. For any three group members i, j, and k, we show that $\beta_j = \beta_k$ if and only if $\Delta Y_j/\Delta Z_i = \Delta Y_k/\Delta Z_i$. The intuition behind this result is that, if two agents j and k share the same peer effect, an exogenous change in Z_i produces identical spillovers on both agents. We can therefore assess whether $\beta_j = \beta_k$ by regressing Y_j and Y_k separately on $(1, Z')'$ and testing whether the coefficients on Z_i are equal for $i \notin \{j, k\}$.

Lemma 1, Part (ii), provides a way to jointly test Assumptions A.1 and A.4. Under these assumptions, the reduced-form effects $\Delta Y_i/\Delta \bar{Z}_{-i}$ and $\Delta \bar{Y}_{-i}/\Delta Z_i$ are equal, since they both correspond to the common spillover effect, $\Delta Y_i/\Delta Z_j$. This property yields a testable implication: the coefficient on $\bar{Z}_{-i}$ in a regression of Y_i on $(1, Z_i, \bar{Z}_{-i})'$ should equal the coefficient on Z_i in a regression of $\bar{Y}_{-i}$ on $(1, Z_i, \bar{Z}_{-i})'$. Rejecting this equality implies that at least one of A.1 and A.4 fails. Note that this second test is particularly straightforward to implement in practice, as it requires only regressions on $(1, Z_i, \bar{Z}_{-i})'$ rather than the full instrument vector.

2.4.2 Frequently Used Linear IV Estimands

Next, we shift attention to linear IV estimands that leverage exclusion restrictions to recover the interaction effect β. This object is often the primary target parameter in classical linear-in-means models, and our framework nests a variety of standard approaches for recovering it.

We analyze a broad class of IV estimands for β that use the vector Z_{-i}, or some monotonic transformation of the vector, as an excluded instrument for $\bar{Y}_{-i}$ in agent i's outcome equation. In particular, we define the instrument as $\tilde{Z}_{-i} = g(Z_{-i})$, where g is a (possibly vector-valued) monotone mapping defined on the support of Z_{-i}. This formulation accommodates a range of IV strategies, including: (1) using one instrument individually, (2) using multiple instruments together, and (3) using an average of multiple instruments.[7] For any $z_i \in \text{supp}(Z_i)$, we define:

$$\beta_i^{\text{IV}}(z_i) = \frac{\text{Cov}(Y_i, \mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})|Z_i = z_i)}{\text{Cov}(\bar{Y}_{-i}, \mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})|Z_i = z_i)}, \tag{8}$$

where $\mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})$ represents the population fitted values from a regression of $\bar{Y}_{-i}$ on $(1, \tilde{Z}_{-i})$.

In the classical linear-in-means model, $\beta_i^{\text{IV}}(z_i)$ point identifies the interaction effect β. In fact, even if Assumptions A.1, A.3, and A.4 fail, this estimand still recovers the agent-specific interaction effect β_i provided that Assumption A.2 holds, so β_i remains fixed across groups. This result, well-established in the literature, yields a testable implication of Assumption A.2:

Lemma 2. Suppose the linear-in-means model has a well-defined reduced form. If Assumption A.2 holds, then, for all $z_i \in \text{supp}(Z_i)$ for which the estimand is well-defined, $\beta_i^{\text{IV}}(z_i) = \beta_i$.

Lemma 2 provides a way to validate Assumption A.2 through overidentifying restrictions. It states that, for any given z_i, every admissible choice of the excluded instrument $\tilde{Z}_{-i}$ should

[7] Formally, we allow any g in the set of functions $\mathcal{G} = \{g : \text{supp}(Z_{-i}) \to \mathbb{R}^n | g(z'_{-i}) \geq g(z_{-i}) \text{ for } z'_{-i} \geq z_{-i}\}$. For instrument relevance to hold, we require that g is strictly increasing in at least one component of Z_{-i}.

recover the same interaction effect β_i. When $N > 2$, distinct components or transformations of Z_{-i} may provide multiple valid instruments for $\bar{Y}_{-i}$. Let $\beta_i^{\text{IV},1}(z_i)$ and $\beta_i^{\text{IV},2}(z_i)$ denote the estimands obtained using two such instruments, $\tilde{Z}_{-i,1}$ and $\tilde{Z}_{-i,2}$, respectively. Assumption A.2 then implies the overidentifying restriction $H_0 : \beta_i^{\text{IV},1} = \beta_i^{\text{IV},2}$. Subject to the maintained validity of the instruments, rejecting this equality implies that Assumption A.2 fails to hold.

Table 2: Economic Inferences from OLS and IV Estimands

Economic Quantity	$\beta^{\text{OLS}}(Y_i;\tilde{Z})$	$\beta^{\text{OLS}}(\bar{Y};\tilde{Z})$	$\beta^{\text{OLS}}(\bar{Y}_{-i};\tilde{Z})$	$N\beta^{\text{OLS}}_{Z_i}(\bar{Y};\tilde{Z})/\beta^{\text{OLS}}_{Z_i}(Y_i;\tilde{Z})$	$\beta_i^{\text{IV}}(z_i)$
Spillover Effect					
Constant Effects	Ind. Effect	—	Ind. Effect	—	—
Heterogeneous Effects	Avg. Effect	—	—	—	—
Total Individual Effect					
Constant Effects	Ind. Effect	—	—	—	—
Heterogeneous Effects	Avg. Effect	—	—	—	—
Total Effect on the Average					
Constant Effects	—	Ind. Effect	—	—	—
Heterogeneous Effects	—	Avg. Effect	—	—	—
Social Multiplier					
Constant Effects	Test: ≤ 1	—	Test: ≤ 1	Ind. Effect	—
Heterogeneous Effects	—	—	Test: ≤ 1 for all g	—	—
Interaction Effect					
Constant Effects	Test: ≤ 0	—	—	—	Ind. Effect
Heterogeneous Effects	Test: ≤ 0 for all g	—	—	—	Weighted Avg.

Notes. The table reviews the identifying content of OLS and linear IV estimands under the constant effects model (Assumptions A.1-A.4) and the heterogeneous effects model. For OLS estimands, we take $\tilde{Z} = (1, Z')'$.

3 Econometric Analysis under Heterogeneous Effects

We now relax Assumptions A.1-A.4 by allowing agents to exhibit interaction effects β_{ig} and direct effects γ_{ig} that vary in both sign and magnitude, within and across groups. Specifically, we treat $\alpha_g = [\alpha_{ig}]_{i\in\mathcal{N}_g}$, $\beta_g = [\beta_{ig}]_{i\in\mathcal{N}_g}$, and $\gamma_g = [\gamma'_{ig}]_{i\in\mathcal{N}_g}$ as random coefficient vectors, jointly distributed according to an unrestricted density f. We allow for arbitrary dependence among these coefficients. For example, an agent's interaction effect can depend on those of her peers or on those of agents in other groups, and her direct response to the shifter Z_{ig} can depend on the unobserved characteristics of other agents. We also allow these coefficients to depend on the group size and composition, as characterized by the set $\mathcal{N}_g$. For example, the amount of social pressure that an agent experiences can be affected by the number or types of peers.[8]

3.1 Economic Quantities under Heterogeneous Effects

We now reinterpret the economic quantities in Table 1 under the heterogeneous effects model. As before, we ease notation by removing group subscripts and treating Z_i as one-dimensional.

[8]Allowing the coefficients to depend on group size ties our hands in the empirical analysis, as it prevents us from using group size variation as a source of identification, as in Lee (2007) and Davezies et al. (2009).

Our first step is to establish a necessary and sufficient condition for a unique equilibrium. Our condition substantially relaxes Assumption A.3. Rather than imposing a bound on the magnitude of interaction effects, we only rule out a knife-edge case under which (1) is singular.

Assumption I (Unique Solution). $\sum_{i=1}^{N}(1-\beta_i)\prod_{j\neq i}(N-1+\beta_j)\neq 0$ with probability one.

Assumption I is a full-rank condition. It requires that $I-\mathbf{B}$ is invertible, ruling out cases in which (1) fails to admit a unique solution. In the two-agent special case ($N=2$), this failure corresponds to outcome equations that form parallel lines: the lines either do not intersect, so the model has no solution, or they coincide, so the model has infinitely many solutions.[9] By ruling out these two cases, Assumption I guarantees that the reduced form is well-defined.

Under Assumption I, we derive a closed-form representation of equilibrium outcomes for heterogeneous interaction effects. Previous work obtains related expressions for special cases with two or three agents (e.g. Masten, 2017). However, our formulas apply to groups of arbitrary size, allowing us to characterize how Z affects Y over a wide range of economic settings.

Proposition 1. Equations (1) possess a unique solution if and only if Assumption I holds. In equilibrium, individual outcomes are characterized by the following reduced form equations:

$$Y_i=\left(1+\beta_i\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}\right)(\alpha_i+\gamma_i Z_i)+\sum_{j\neq i}\psi_{ij}(\alpha_j+\gamma_j Z_j),\quad \text{for } i\in\{1,\dots,N\},\tag{9}$$

where ψ_{ij} represents the spillover effect agent j's outcome Y_j on agent i's outcome Y_i, equal to:

$$\psi_{ij}=\beta_i\times\frac{\prod_{\ell\notin\{i,j\}}\left(1+\frac{\beta_\ell}{N-1}\right)}{(N-1)\times\det(I-\mathbf{B})}.$$

The determinant of $I-\mathbf{B}$ has a closed-form representation, which is derived in the Appendix.[10]

Spillover Effect. We now reinterpret the spillover effect of Z_j on Y_i under the heterogeneous effects model. Unlike in the classical model, this quantity can vary across agent pairs (i,j), reflecting heterogeneity in direct effects, γ_j, of Z_j on Y_j and spillover effects, ψ_{ij}, of Y_j on Y_i:

$$\frac{\Delta Y_i}{\Delta Z_j}=\gamma_j\times\psi_{ij}.\tag{10}$$

To understand how $\Delta Y_i/\Delta Z_j$ depends on the interaction effects β_i, it is useful to introduce a boundedness condition, requiring that the interactions are sufficiently small in magnitude:

[9] Tamer (2003) studies both *incoherency* and *incompleteness* of simultaneous equation models. When the model is incoherent, it has no solution. When the model is incomplete, it has multiple solutions. In our setup, nonintersecting lines imply the model is incoherent, while overlapping lines imply the model is incomplete.

[10] In our formulas, we take the empty product to equal one. So, we define $\psi_{ij}=\frac{\beta_i}{(N-1)\det(I-\mathbf{B})}$ for $N=2$.

Assumption II (Bounded Interactions)**.** $1-N<\beta_i<1$ for all i with probability one.

Assumption II guarantees that the determinant of I–$\mathbf{B}$, as well as the product terms entering ψ_{ij}, are strictly positive with probability one. Hence, ψ_{ij} has the same sign as β_i. As a result, when $\gamma_j>0$, the spillover effect $\Delta Y_i/\Delta Z_j$ inherits the sign of agent i's interaction effect β_i.

This restriction has a natural interpretation in our economic applications. For example, in the peer effects model, when student i prefers to conform to her peers ($\beta_i>0$), a positive shock to student j's ability raises i's achievement. In the household labor-supply model, when individual i values leisure ($\mu_i>0$), an increase in member j's wage reduces i's labor supply. In the oligopoly model, when demand is downward-sloping ($b>0$), a favorable productivity shock to a rival firm j reduces firm i's output. Note that Assumption II is a special case of Assumption I, and is therefore a sufficient condition for the reduced form to be well-defined. Although this assumption facilitates economic interpretation, our main identification results do not rely on it. Hence, we do not impose Assumption II throughout most of our analysis.[11]

Total Individual Effect. Next, we reinterpret the total individual effect of Z_i on Y_i. Under heterogeneous interactions, this quantity can vary across individuals i, reflecting differences in the direct effects, γ_i, of Z_i on Y_i and the indirect feedback effects, $\beta_i(\Delta\bar{Y}_{-i}/\Delta Z_i)$. We write:

$$\frac{\Delta Y_i}{\Delta Z_i}=\gamma_i+\underbrace{\beta_i\frac{\Delta\bar{Y}_{-i}}{\Delta Z_i}}_{\text{Indirect Effect}},\quad\text{where}\quad\frac{\Delta\bar{Y}_{-i}}{\Delta Z_i}=\gamma_i\times\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}. \tag{11}$$

Under Assumption II, the total individual effect has the same sign as γ_i. Therefore, if Z_i has a positive direct effect on Y_i, it also has a positive effect on Y_i in equilibrium. For example, in the peer effects model, a student's achievement does not fall when her ability rises. In the household labor-supply model, a person's income does not fall after an increase in her own wage. In an oligopoly model, a firm's output does not fall after a favorable productivity shock.

Unlike in the classical linear-in-means model, heterogeneous interactions can either amplify or attenuate the equilibrium effect of Z_i on Y_i relative to γ_i. This relationship is governed by the sign of the feedback term, $\beta_i\times\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$, which aggregates all the interaction paths that begin and end with agent i. A positive feedback term reinforces the direct response, so $|\Delta Y_i/\Delta Z_i|>|\gamma_i|$. Meanwhile, a negative term offsets the direct response, so $|\Delta Y_i/\Delta Z_i|<|\gamma_i|$.

Under Assumption II, ψ_{ji} has the same sign as β_j. Hence, if all interaction effects share a common sign, the feedback term $\beta_i\times\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$ is positive. Models with uniformly positive interactions, as in standard peer effects applications, or uniformly negative interactions, as in our household labor-supply and oligopoly examples, therefore deliver self-reinforcing feedback that amplifies an agent i's response to an own shock. Models with mixed-sign interactions,

[11] In the household labor supply model, Assumption II holds if all individuals value consumption: $\mu_i\neq 1$ for all i. In the oligopoly model, it rules out Bertrand competition for firms with constant marginal costs: $(\theta_i,\delta_i)\neq(-1,0)$ for all i. Such models have no interior solution, as firms always seek to undercut one another until they are left with zero profit. This phenomenon is known as the Bertrand paradox (Edgeworth, 1925).

such as one where an agent i differentiates ($\beta_i < 0$) while her peers conform ($\beta_j > 0$ for $j \neq i$), instead generate self-offsetting feedback that attenuates agent i's response to an own shock.

Total Effect on the Average. Next, we reexamine the total effect on the average, $\Delta\bar{Y}/\Delta Z_i$. Unlike in the classical model, this quantity can vary across agents i, since the same shock can have different effects on group-level outcomes depending on which agent is directly exposed:

$$\frac{\Delta\bar{Y}}{\Delta Z_i} = \frac{\prod_{\ell\neq i}\left(1+\frac{\beta_\ell}{N-1}\right)}{N \times \det(I-\mathbf{B})} \times \gamma_i. \tag{12}$$

Under Assumption II, the term multiplying γ_i is strictly positive, so $\Delta\bar{Y}/\Delta Z_i$ has the same sign as γ_i. Thus, in a peer effects model, a policy that directly raises one student's achievement also raises average achievement in the peer group. In a household labor-supply model, an increase in a single household member's wage also raises average household income. In an oligopoly model, a favorable productivity shock to one firm raises average market output.

Social Multiplier Effect. We now turn to the social multiplier effect $(\Delta\bar{Y}/\Delta\bar{Z})/(\Delta Y_i/\Delta Z_i)$, as defined by Glaeser et al. (2003). With heterogeneous interactions, this quantity is no longer well-defined, since the total effect of an exogenous shock on the group average depends on which agent(s) are directly exposed. As a result, any given change in $\bar{Z}$ can arise from different perturbations to the vector $(Z_1, \ldots, Z_N)$, each of which may have different impacts on $\bar{Y}$.

For models that involve heterogeneous interactions, we define an "agent-specific" social multiplier, which compares the total effect of Z_i on $\bar{Y}$ with the individual effect of Z_i on Y_i:

$$M_{(i)} = \frac{\sum_{j=1}^{N}\Delta Y_j/\Delta Z_i}{\Delta Y_i/\Delta Z_i} = \frac{1}{1-\frac{1}{N-1}\sum_{j\neq i}\frac{\beta_j}{1+\beta_j/(N-1)}}. \tag{13}$$

This definition extends the classical multiplier to settings with heterogeneity. Under Assumptions A.1-A.4, $M_{(i)}$ reduces to M for all i. As N grows large, $M_{(i)}$ tends to $(1-\frac{1}{N-1}\sum_{j\neq i}\beta_j)^{-1}$.

The agent-specific social multiplier $M_{(i)}$ has a natural interpretation if group members assume distinct roles. For example, consider a household labor-supply model in which agent i is the primary earner and agent j is the secondary earner. In our framework, this distinction reflects that j places greater value on household and other nonmarket activities, so $\mu_j > \mu_i$. In a two-person household, the agent-specific multipliers are $M_{(i)} = 1-\mu_j$ and $M_{(j)} = 1-\mu_i$. Hence, agent i has a smaller multiplier effect than agent j. This difference reflects that a wage increase for the primary earner i induces a relatively large reduction in the secondary earner j's labor supply, offsetting more of i's initial income gain. The household-income effect of a wage shock therefore depends on whether it is received by the primary or secondary earner.[12]

Two natural aggregate summaries follow from our definition in (13). The first is an aver-

[12] In our household labor supply example, multiplier effects are always less than or equal to one ($M_{(i)} \leq 1$ for all i), because added-earner responses always partially offset the income gain from an own-wage shock.

age of agent-specific multiplier effects, $M^{\text{avg}} = \frac{1}{N}\sum_{i=1}^{N} M_{(i)}$, which gives equal weight to the multiplier associated with each agent. The second is the ratio of average effects, comparing the effect of a common unit shift in all Z_i on the group average with the average own effect:

$$M = \frac{\sum_{i=1}^{N} \Delta\bar{Y}/\Delta Z_i}{\frac{1}{N}\sum_{j=1}^{N} \Delta Y_j/\Delta Z_j} = \frac{\frac{1}{N}\sum_{j=1}^{N}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\gamma_j}{\frac{1}{N}\sum_{j=1}^{N}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\left(1-\frac{1}{N-1}\sum_{k\neq j}\frac{\beta_k}{1+\beta_k/(N-1)}\right)\gamma_j}. \tag{14}$$

Although these two aggregations can differ under heterogeneous effects, both reduce to the original social multiplier in (6) under the classical linear-in-means assumptions. We use (14) as our definition of the aggregate social multiplier throughout the remainder of the paper. If γ_i is constant across agents i, then M tends to $\left(1-\frac{1}{N}\sum_{i=1}^{N}\beta_i\right)^{-1}$ for large group sizes N.

3.2 Analysis of OLS and IV under Heterogeneous Effects

We now examine what can and cannot be learned from frequently used OLS and IV estimands for linear-in-means models under heterogeneous effects. We show that, while these estimands no longer deliver point identification, they are still informative about key economic quantities.

To analyze these estimands under heterogeneous effects, we replace the usual instrument exogeneity condition with instrument independence, $Z \perp (\alpha, \beta, \gamma, \mathcal{N})$, requiring the shifters to be independent of the structural coefficients and group structure. Also, as in Section 2.4, we take each Z_i to be an individual-level shifter and assume that $\mathbb{E}[ZZ']$ is nonsingular. For instrument relevance, we further assume $\mathrm{P}(\gamma_i \neq 0) > 0$ for all i. Standard in random coefficient models, our independence assumption can be weakened to independence conditional on observed agent or group characteristics, such as group size or composition.[13] Our relevance assumption can also be relaxed to allow Z to enter only a subset of agents' outcome equations.

Since we focus on mean-based identification strategies, an additional requirement for our analysis is that the relevant reduced-form coefficients have finite moments. While Assumption I guarantees that the reduced form is well-defined by ruling out cases where $\det(I-\mathbf{B})=0$, it does not eliminate cases where $\det(I-\mathbf{B})$ is close to zero with high probability. Hence, entries of $(I-\mathbf{B})^{-1}$ may have undefined moments even though the reduced form exists almost surely. To ensure that the estimands we study are well-defined, we invoke a slightly stronger integrability condition, which we derive by adapting Assumption A6 of Masten (2017) to our setting:

Assumption III (Sufficient Conditions for Moment Determinacy)**.**

III.1 $\mathrm{P}\left(\left|\sum_{i\in\mathcal{N}}(1-\beta_i)\prod_{j\in\mathcal{N}\setminus i}(|\mathcal{N}|-1+\beta_j)\right| \geq \tau\right) = 1$ for some scalar $\tau > 0$.

III.2 The marginal distributions of α_i and γ_i have subexponential tails.

Assumption III.1 bounds the denominator of the reduced-form coefficients away from zero. Meanwhile, Assumption III.2 controls the tails of the structural coefficients, implying that the

[13] If Z contains shifters Z^s and covariates Z^c, then the condition can be weakened to $Z^s \perp (\alpha, \beta, \gamma, \mathcal{N}) \mid Z^c$. Dependence on group structure can likewise be accommodated by conditioning on group size or composition.

distribution of Y does not place excessive probability mass on extreme outcomes. Together, these two conditions are sufficient to ensure that the reduced-form parameters we study have finite moments. Necessary conditions for moment determinacy are studied by Masten (2017).

3.2.1 Empirical Analysis of OLS Estimands

We begin by examining the identifying content of OLS estimands under heterogeneous effects. We consider three types of estimands: projections of Y on the full instrument vector $(1, Z')'$, projections of Y on $(1, Z_i, \bar{Z}_{-i})'$, and OLS coefficient ratios used to recover social multipliers.

OLS with the Full Instrument Vector. Consider the OLS estimands in Table 2, obtained by projecting Y_i, $\bar{Y}$, and $\bar{Y}_{-i}$ on the instrument vector $(1, Z')'$. In a classical linear-in-means model, these estimands identify the (homogeneous) reduced form coefficients. Under heterogeneous effects, they instead recover unweighted averages of these coefficients across groups.

Proposition 2. In the linear-in-means model with heterogeneous interactions, OLS recovers:

(i) The average spillover effect: $\beta^{\text{OLS}}_{Z_j}(Y_i; (1, Z')') = \text{E}(\Delta Y_i/\Delta Z_j)$ for all $i \neq j$.

(ii) The average total individual effect: $\beta^{\text{OLS}}_{Z_i}(Y_i; (1, Z')') = \text{E}(\Delta Y_i/\Delta Z_i)$ for all i.

(iii) The average total effect on the average: $\beta^{\text{OLS}}_{Z_i}(\bar{Y}; (1, Z')') = \text{E}(\Delta \bar{Y}/\Delta Z_i)$ for all i.

Proposition 2 demonstrates that, even in settings with heterogeneous interactions, OLS projections of outcomes on the instrument vector $(1, Z')'$ remain informative about equilibrium responses to individual-level shocks. For example, in a peer effects model, these OLS regressions identify how student achievement responds, on average across classrooms, to student-level interventions. In a household labor-supply model, they identify how earnings respond, on average across households, to individual wage shocks. In an oligopoly model, they identify how firm output responds, on average across markets, to firm-specific productivity shocks.

OLS with Own and Peer-Average Shifters. We next consider OLS estimands obtained by projecting Y_i, $\bar{Y}$, and $\bar{Y}_{-i}$ on $(1, Z_i, \bar{Z}_{-i})'$. These regressions are typically more convenient to implement in practice, as they replace the full vector of peer shifters with its average, thus reducing the number of regressors. In the classical linear-in-means model, this aggregation is without loss: homogeneity implies that all peer shifters enter the reduced form with the same coefficient, so $\Delta Y_i/\Delta Z_j = \Delta Y_i/\Delta Z_k$ for all $i \neq j \neq k$. The resulting estimands therefore still identify the spillover effect, the total individual effect, and the total effect on the average.

This equivalence fails with heterogeneous interactions. When β_i and γ_i vary across agents, the reduced form effects of individual peer shifters differ, so $\Delta Y_i/\Delta Z_j \neq \Delta Y_i/\Delta Z_k$ for $i \neq j \neq k$. Replacing $\{Z_j\}_{j \neq i}$ with $\bar{Z}_{-i}$ then imposes a common coefficient on variables that have heterogeneous effects. Consequently, an OLS regression that includes averages of Z, while excluding $\{Z_j\}_{j=1}^N$ as individual regressors, suffers from omitted variable bias.[14] To recover unweighted

[14]This bias arises even when the parameters β_i and γ_i are constant across groups, but vary within groups.

averages of reduced form quantities of interest under heterogeneous interactions, we therefore emphasize that it is important to include each individual shifter, Z_j, as a separate regressor.

OLS Estimands for Social Multipliers. Lastly, we study a class of OLS-based estimands that are commonly used to recover social multipliers. In the classical linear-in-means model, the multiplier is identified from a ratio of OLS coefficients, which may be computed either as:

$$M_{(i)}^{\mathrm{OLS}} = N \frac{\beta_{Z_i}^{\mathrm{OLS}}(\bar{Y}; (1, Z')')}{\beta_{Z_i}^{\mathrm{OLS}}(Y_i; (1, Z')')},$$

for any given agent i, or instead as a ratio of OLS coefficients averaged across group members:

$$M^{\mathrm{OLS}} = N \frac{\sum_{i=1}^{N} \beta_{Z_i}^{\mathrm{OLS}}(\bar{Y}; (1, Z')')}{\sum_{i=1}^{N} \beta_{Z_i}^{\mathrm{OLS}}(Y_i; (1, Z')')}.$$

Under Assumptions A.1-A.4, these two ratios coincide. More generally, when β_i and γ_i vary within groups but are fixed across groups, the first ratio recovers the agent-specific multiplier $M_{(i)}$ defined in (13), and the second ratio recovers the aggregate multiplier M defined in (14).

These interpretations break down when the structural coefficients vary across groups. In that case, OLS recovers average reduced-form effects, so the coefficient ratios correspond to:

$$M_{(i)}^{\mathrm{OLS}} = \frac{\sum_{j=1}^{N} \mathrm{E}(\Delta Y_j / \Delta Z_i)}{\mathrm{E}(\Delta Y_i / \Delta Z_i)} \quad \text{and} \quad M^{\mathrm{OLS}} = \frac{\sum_{i=1}^{N} \mathrm{E}(\Delta \bar{Y} / \Delta Z_i)}{\frac{1}{N} \sum_{j=1}^{N} \mathrm{E}(\Delta Y_j / \Delta Z_j)}.$$

These estimands represent ratios of average equilibrium effects, not averages of group-specific multipliers. They therefore do not point identify $M_{(i)}$, M, or their population means. As we show in Section 3.3, OLS estimands can still yield informative bounds on multiplier effects; however, standard coefficient-ratio formulas do not carry over to settings with heterogeneity.

3.2.2 Empirical Analysis of IV Estimands

We now reexamine the IV estimand in (8). Under heterogeneous interactions, this estimand no longer point identifies the interaction effect β_i. This negative result motivates our analysis of when IV remains informative and what it can tell us about the distribution of peer effects.

Our analysis proceeds in two parts. First, we derive conditions under which IV estimands can be represented as positively weighted averages of β_i. We view this condition as a minimal requirement for the estimand to be informative about peer effects. Next, using this weighted average representation, we provide a new economic interpretation of the IV estimand, characterizing which features of the distribution of interaction effects IV recovers. In Section 3.3, we use this characterization to derive bounds on the unweighted average of interaction effects.

In this setting, if $\{Z_j\}_{j=1}^{N}$ are all uncorrelated, the coefficient on Z_i in a regression of Y_i on $(1, Z_i, \bar{Z}_{-i})$ would still recover the total individual effect $\Delta Y_i / \Delta Z_i$. Yet, the other coefficients would be biased by construction.

Conventional Conditions for Positive Weights. A standard sufficient condition for the IV estimand to have a positive weighted-average interpretation is the monotonicity condition of Imbens & Angrist (1994), widely adopted in the treatment-effects literature. It maintains that the endogenous variable $\bar{Y}_{-i}$ is affected uniformly by any change in the instrument $\tilde{Z}_{-i}$. When $\tilde{Z}_{-i} = Z_{-i}$ (or is a one-to-one transformation of Z_{-i}), this condition can be stated as:

Assumption IAM (Imbens-Angrist Monotonicity)**.** For any realizations (z_{-i}, z_i) and (z'_{-i}, z_i) of the vector Z, either $\mathrm{P}\left(\bar{Y}_{-i}(z_{-i}, z_i) \geq \bar{Y}_{-i}(z'_{-i}, z_i)\right) = 1$ or $\mathrm{P}\left(\bar{Y}_{-i}(z_{-i}, z_i) \leq \bar{Y}_{-i}(z'_{-i}, z_i)\right) = 1$.

We argue that this condition is plausible in two-agent settings but unlikely to hold in groups of three or more. To formalize these arguments, we analyze models with two and three agents.

Two-Agent Groups $(N = 2)$. Consider a model in which each group contains only two agents:

$$\begin{aligned} Y_1 &= \alpha_1 + \beta_1 Y_2 + \gamma_1 Z_1 \\ Y_2 &= \alpha_2 + \beta_2 Y_1 + \gamma_2 Z_2. \end{aligned}$$

This special case allows us to study peer effects between pairs of students, joint labor supply decisions within two-person households, and strategic interactions among firms in duopolies.

For any $j \neq i$, the IV estimand equals $\beta_i^{IV}(z_i) = \mathrm{Cov}(Y_i, Z_j | Z_i = z_i) / \mathrm{Cov}(Y_j, Z_j | Z_i = z_i)$. This estimand can be expressed as a weighted average of interaction effects β_i across groups:

$$\beta_i^{IV}(z_i) = \int_{\mathrm{supp}(\beta_i)} b_i \times \omega(b_i) db_i, \quad \text{where} \quad \omega(b_i) = \frac{\mathrm{E}(\Delta Y_j / \Delta Z_j | \beta_i = b_i) f_{\beta_i}(b_i)}{\mathrm{E}(\Delta Y_j / \Delta Z_j)}. \tag{15}$$

A sufficient condition for the weights to be non-negative, $\omega(b_i) \geq 0$, is the standard Imbens-Angrist monotonicity assumption, which requires Z_j to shift Y_j in the same direction across all groups. In the two-agent model, this assumption holds if and only if γ_j has a common sign across groups, so the direct effect of Z_j on Y_j is uniformly positive (or uniformly negative):

$$\mathrm{P}(\gamma_j \geq 0) = 1 \text{ or } \mathrm{P}(\gamma_j \leq 0) = 1. \tag{16}$$

Under this restriction, the IV estimand is a positively weighted average of interaction effects, assigning greater weight to groups in which Z_j has a larger effect on Y_j. Note that this positive weighted average representation requires no restrictions on the interaction effects β_1 and β_2.

The weighted average interpretation is meaningful in each of our economic applications. First, consider a two-student peer effects model, where Z_j indicates receipt of a scholarship that uniformly raises student j's achievement. Here, $\beta_i^{IV}(z_i)$ recovers a weighted average of i's peer effect β_i, placing greater weight on pairs where the scholarship has a larger impact on j's achievement. Second, consider a two-person household labor-supply model, where Z_j is a policy that uniformly raises member j's wage. Here, $\beta_i^{IV}(z_i)$ recovers a weighted average of i's added-earner effect, placing greater weight on households where j's earnings respond more strongly to the policy. Third, consider a duopoly model, where Z_j is a cost shock that uniformly raises firm j's output. Here, $\beta_i^{IV}(z_i)$ recovers a weighted average of firm i's conduct parameter, placing more weight on markets where j's output is more responsive to the shock.

Three-Agent Groups ($N = 3$). We now consider a three-agent version of the model, given by:

$$\begin{aligned}
Y_1 &= \alpha_1 + \beta_1\Big(\frac{Y_2 + Y_3}{2}\Big) + \gamma_1 Z_1 \\
Y_2 &= \alpha_2 + \beta_2\Big(\frac{Y_1 + Y_3}{2}\Big) + \gamma_2 Z_2 \\
Y_3 &= \alpha_3 + \beta_3\Big(\frac{Y_1 + Y_2}{2}\Big) + \gamma_3 Z_3.
\end{aligned}$$

This case allows us to study peer effects among triads of students, joint labor supply decisions in three-person households, and strategic interactions among firms in three-firm oligopolies.

In this model, the endogenous regressor in agent i's outcome equation is $\bar{Y}_{-i} = (Y_j + Y_k)/2$. A researcher may use either Z_j or Z_k as an excluded instrument for $\bar{Y}_{-i}$. We focus on an IV strategy that uses both instruments jointly, $\tilde{Z}_{-i} = (Z_j, Z_k)'$, although our results extend to IV strategies where monotone transformations of (Z_j, Z_k) are used as excluded instruments.

As in the two-agent case, we can write $\beta_i^{\mathrm{IV}}(z_i)$ as a weighted average of interaction effects, where larger weights are given to values of β_i in groups where $\bar{Y}_{-i}$ is more affected by (Z_j, Z_k):

$$\beta_i^{IV}(z_i) = \int_{\mathrm{supp}(\beta_i)} b_i \times \omega(b_i|z_i) db_i,$$

$$\text{where} \quad \omega(b_i|z_i) = \frac{\mathrm{E}\Big(\frac{\Delta\bar{Y}_{-i}}{\Delta Z_j}\Big|\beta_i{=}b_i\Big)\,\mathrm{Cov}(\bar{Y}_{-i},Z_j|z_i)+\mathrm{E}\Big(\frac{\Delta\bar{Y}_{-i}}{\Delta Z_k}\Big|\beta_i{=}b_i\Big)\,\mathrm{Cov}(\bar{Y}_{-i},Z_k|z_i)}{\mathrm{E}\Big(\frac{\Delta\bar{Y}_{-i}}{\Delta Z_j}\Big)\,\mathrm{Cov}(\bar{Y}_{-i},Z_j|z_i)+\mathrm{E}\Big(\frac{\Delta\bar{Y}_{-i}}{\Delta Z_k}\Big)\,\mathrm{Cov}(\bar{Y}_{-i},Z_k|z_i)} f_{\beta_i}(b_i).$$

The standard condition for these weights to be non-negative is Imbens-Angrist monotonicity. In the three-agent model, this condition requires that every admissible change in the vector (Z_j, Z_k) moves $\bar{Y}_{-i}$ in the same direction across all groups. Unlike in the two-agent case, this condition does not simply involve a sign restriction on the first stage; it also restricts the relative strength of first stage effects across groups. Figure 1 illustrates the implied restrictions of Imbens-Angrist monotonicity for $\Delta\bar{Y}_{-i}/\Delta Z_j$ and $\Delta\bar{Y}_{-i}/\Delta Z_k$. With binary instruments, it requires these effects to have fixed signs across groups and imposes a common ordering on their magnitudes: the same first stage effect must be weakly larger in all groups. With continuous instruments, it further requires the ratio $(\Delta\bar{Y}_{-i}/\Delta Z_j)/(\Delta\bar{Y}_{-i}/\Delta Z_k)$ to be fixed across groups.

We next provide a structural interpretation of the Imbens-Angrist monotonicity assumption in the three-agent model, first focusing on the case in which both instruments are binary, and then turning to the case with continuous instruments. In both cases, we find that, once groups contain more than two agents, monotonicity imposes substantive restrictions on the endogenous interaction effects. We argue that these restrictions are hard to justify in practice.

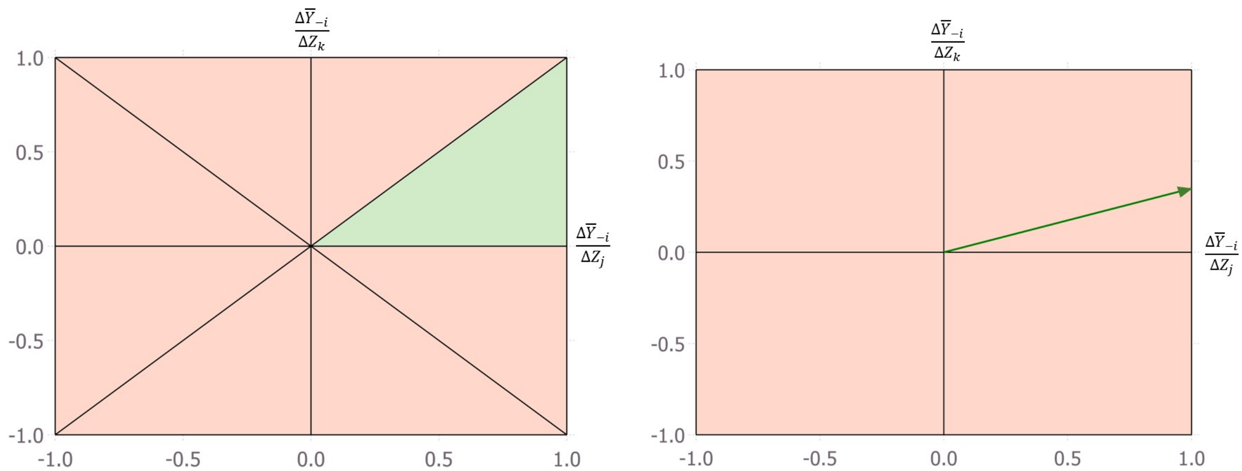
Lemma 3. When $N = 3$ and (Z_j, Z_k) are binary, Assumption IAM holds if and only if:

(i) $\mathrm{P}(\gamma_\ell \geq 0) = 1$ or $\mathrm{P}(\gamma_\ell \leq 0) = 1$, for $\ell \in \{j, k\}$.

(ii) $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \geq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$ or $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \leq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$.

To better interpret the economic content of Assumption IAM in the binary-instrument case, suppose the direct effects are constant, $\gamma_j = \gamma_k = \gamma$, and consider our three economic examples. In a peer-effects model, IAM requires a common ordering of peer effects: either $\Pr(\beta_j \geq \beta_k) = 1$ or $\Pr(\beta_j \leq \beta_k) = 1$. As these peer effects are precisely the quantities to be estimated, it would be unnatural in most settings to impose this structure up front. In a household labor-supply model, IAM requires a common ordering of leisure valuations, either $\Pr(\mu_j \geq \mu_k) = 1$ or $\Pr(\mu_j \leq \mu_k) = 1$, which amounts to assuming a priori which household member has a stronger preference for nonmarket time. In an oligopoly model, IAM requires a common ordering of the conduct-cost index: either $\delta_j + b\theta_j$ is always above $\delta_k + b\theta_k$ or always below it. Since this index combines firms' marginal-cost curvature with their conjectured effects on market prices, such an ordering is hard to justify unless agents are labeled to satisfy it by construction.

Figure 1: Illustration of IAM Restrictions for Three-Agent Groups



(a) Case 1: Binary Z_j and Z_k

(b) Case 2: Continuous Z_j and Z_k

Notes. These plots show feasible regions of the reduced form effects $\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j}$ and $\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k}$ under Assumption IAM.

With continuous instruments, Imbens-Angrist monotonicity imposes an even stronger restriction on the endogenous interaction effects. The next result makes this restriction explicit.

Lemma 4. When $N = 3$ and (Z_j, Z_k) are continuous, Assumption IAM holds if and only if:

(i) $\mathrm{P}(\gamma_\ell \geq 0) = 1$ or $\mathrm{P}(\gamma_\ell \leq 0) = 1$, for $\ell \in \{j, k\}$.

(ii) $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \geq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$ or $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \leq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$.

As before, we interpret this lemma using our three economic examples under the simplifying assumption of common direct effects: $\gamma_j = \gamma_k = \gamma$. In a peer-effects model, IAM imposes a deterministic linear relationship between the peer effects: $\beta_j = 2(a-1) + a\beta_k$ for some fixed $a \in \mathbb{R}$. In a household labor-supply model, it imposes that household members' consumption-leisure preferences are deterministic linear functions of one another: $\frac{2-\mu_j}{1-\mu_j} = a \times \frac{2-\mu_k}{1-\mu_k}$ for some fixed $a \in \mathbb{R}$. In an oligopoly model, it imposes an analogous restriction on firms' conduct-cost indices: $(\delta_j + b\theta_j) = a \times (\delta_k + b\theta_k) + 1.5b(a-1)$. These restrictions appear to have no obvious primitive justification. Therefore, in empirical settings with heterogeneous interaction effects and multiple continuous instruments, we view Assumption IAM as being difficult to justify.

Alternative Conditions for Positive Weights. Motivated by the finding that standard monotonicity conditions entail strong assumptions on the endogenous interaction effects, we propose an alternative condition under which the IV estimand admits a positively weighted-average interpretation. Our condition follows Mogstad et al. (2021) by imposing monotonicity instrument by instrument. In particular, rather than requiring that $\bar{Y}_{-i}$ moves in a common direction for every change in the vector Z_{-i}, we require only that the individual components of Z_{-i}, holding others fixed, move $\bar{Y}_{-i}$ in a common direction. We state this condition below:

Assumption PM (Partial Monotonicity)**.** For any $j \neq i$ and any (z_j, z_{-j}) and (z'_j, z_{-j}) in the support of Z, either $\mathrm{P}\left(\bar{Y}_{-i}(z_j, z_{-j}) \geq \bar{Y}_{-i}(z'_j, z_{-j})\right) = 1$ or $\mathrm{P}\left(\bar{Y}_{-i}(z_j, z_{-j}) \leq \bar{Y}_{-i}(z'_j, z_{-j})\right) = 1$.

This condition is strictly weaker than IAM. Moreover, as we demonstrate in the next lemma, it imposes no restrictions on the endogenous interaction effects, requiring only that, for each shifter Z_j, the direct effect of Z_j on agent j's outcome Y_j has a common sign across groups.

Lemma 5. Assumption PM holds if and only if $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$ for all $j \neq i$.

Because partial monotonicity leaves the interaction effects unrestricted, we view it as a more attractive alternative to IAM in most applications of the model. We now show how IAM can be replaced by partial monotonicity while preserving the positive weighted-average interpretation of the IV estimand. The remaining requirement is a condition on the instrument correlation structure. This condition, stated below, restricts the covariances of the peer shifters, Z_j, relative to their average reduced-form effects on the endogenous regressor, $\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_j)$.

Assumption NNW (No Negative Weights)**.** Fix some $z_i \in \mathrm{supp}(Z_i)$. For any $j, k \in \mathcal{N} \setminus i$:

$$\mathrm{Cov}(Z_j, Z_k|z_i) \notin \left(-\sum_{\ell \notin \{i,j\}} \frac{\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_\ell)}{\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_j)} \mathrm{Cov}(Z_\ell, Z_k|z_i), -\sum_{\ell \notin \{i,k\}} \frac{\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_\ell)}{\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_k)} \mathrm{Cov}(Z_\ell, Z_j|z_i) \right).$$

All objects entering Assumption NNW are identified in the data: the instrument covariances are observed, and the average reduced-form effects are OLS regression coefficients. NNW can therefore be validated empirically, rather than relying on untestable economic assumptions.

To better interpret assumption NNW, we give two useful special cases. First, NNW holds automatically when the peer shifters Z_{-i} are mutually uncorrelated. Second, if the components of γ_{-i} share the same sign, it holds when no two instruments are negatively correlated.

Lemma 6. Assumption NNW is satisfied if either: *(1)* $\text{Cov}(Z_j, Z_k|z_i) = 0$ for all $j, k \in \mathcal{N} \setminus i$ or if *(2)* both $\text{Cov}(Z_j, Z_k|z_i) \geq 0$ for all $j, k \in \mathcal{N} \setminus i$ and $\text{P}(\gamma_{-i} \geq 0) = 1$ or $\text{P}(\gamma_{-i} \leq 0) = 1$.

Together, Assumptions PM and NNW are sufficient to guarantee that the IV estimand is a positively weighted average of interaction effects. Our next result states this conclusion for a general case in which the excluded instrument can be any subvector of Z_{-i}, or any one-to-one transformation of such a subvector. It also gives sufficient conditions under which the same interpretation extends to non-injective transformations, such as the average of peer shifters.

Proposition 3. Choose $\tilde{Z}_{-i} \subseteq Z_{-i}$, and suppose that Assumptions PM and NNW both hold. Then the IV estimand is a positively-weighted average of instrument-specific IV estimands:

$$\beta_i^{\text{IV}}(z_i) = \sum_{j \neq i} \omega_j \times \frac{\text{Cov}(Y_i, Z_j|z_i)}{\text{Cov}(\bar{Y}_{-i}, Z_j|z_i)}, \quad \text{where: } \sum_{j \neq i} w_j = 1 \text{ and } w_j \geq 0, \ \forall j \neq i.$$

Additionally, the IV estimand represents a positively-weighted average of interaction effects:

$$\beta_i^{\text{IV}}(z_i) = \int_{\text{supp}(\beta_i)} \beta_i \times \omega(\beta_i|z_i) d\beta_i, \quad \text{where: } \int \omega(\beta_i|z_i) d\beta_i = 1 \text{ and } \omega(\beta_i|z_i) \geq 0, \ \forall \beta_i.$$

For a non-injective transformation $\tilde{Z}_{-i} = g(Z_S)$ of a subvector $Z_S = (Z_j)_{j \in S}$, the same conclusion applies if: (i) $\text{P}(\gamma_{-i} \geq 0) = 1$ or $\text{P}(\gamma_{-i} \leq 0) = 1$ and (ii) $\text{Cov}(Z_j, Z_k|z_i) \geq 0$, for any $j, k \neq i$:

To interpret Proposition 3, we now reconsider the special case of the model with three agents.

Three-Agent Groups ($N = 3$). Suppose there are three agents, $\{i, j, k\} \in \{1, 2, 3\}$. Assumption PM then requires γ_j and γ_k to be uniformly signed across groups, while Assumption NNW reduces to a restriction that the peer shifters, Z_j and Z_k, cannot be too negatively correlated:

$$\text{Cov}(Z_j, Z_k \mid z_i) \notin \left(-\frac{\text{E}(\Delta \bar{Y}_{-i}/\Delta Z_j)}{\text{E}(\Delta \bar{Y}_{-i}/\Delta Z_k)} \text{Var}(Z_j \mid z_i), -\frac{\text{E}(\Delta \bar{Y}_{-i}/\Delta Z_k)}{\text{E}(\Delta \bar{Y}_{-i}/\Delta Z_j)} \text{Var}(Z_k \mid z_i) \right). \tag{17}$$

Under these assumptions, the IV estimand has a direct interpretation in our economic examples. In a peer effects model, it is a positively weighted average of peer effects β_i, with greater weight on groups where the achievement of students j and k is most responsive to $\tilde{Z}_{-i}$. In a household labor-supply model, it is a positively weighted average of member i's added-earner effect, with greater weight on households where the earnings of j and k are most responsive to $\tilde{Z}_{-i}$. In an oligopoly model, it is a positively weighted average of firm i's conduct parameter, with greater weight on markets in which the output of j and k is most responsive to $\tilde{Z}_{-i}$.

Conditional IV with One Instrument. In cases where Assumptions NNW and PM fail, a more targeted IV specification can still admit a positively weighted-average interpretation. In particular, consider an IV estimand that uses only one shifter Z_j as the excluded instrument, while controlling for the remaining shifters Z_{-j}. We define this estimand in the following way:

$$\beta_i^{IV}(z_{-j}) = \frac{\text{Cov}(Y_i, Z_j | Z_{-j} = z_{-j})}{\text{Cov}(\bar{Y}_{-i}, Z_j | Z_{-j} = z_{-j})} \tag{18}$$

To interpret this IV estimand as a positively-weighted average of interaction effects, we only require that $\Delta\bar{Y}_{-i}/\Delta Z_j$ has a common sign across groups. This condition imposes the same parametric restriction as IAM monotonicity in the $N = 2$ case, namely that $\text{P}(\gamma_j \geq 0) = 1$ or $\text{P}(\gamma_j \leq 0) = 1$. Under this condition, $\beta_i^{IV}(z_{-j})$ is a positively-weighted average of interaction effects, with larger weights placed on values of β_i in groups where $\bar{Y}_{-i}$ is most responsive to Z_j.

3.3 Bounding Peer Effects and Multipliers under Heterogeneity

In this subsection, we explain how to use OLS and linear IV estimands to learn about endogenous interactions and social multipliers in linear-in-means models with heterogeneous effects.

3.3.1 Using IV to Bound Average Peer Effects

We first show how linear IV estimands can be used to bound unweighted averages of endogenous interaction effects. The key step is to characterize economic environments in which the IV estimand lies above or below $\text{E}(\beta_i)$. The following proposition provides this classification.

Proposition 4. Suppose $\beta_i^{\text{IV}}(z_i)$ is a positively weighted average of β_i, and let β_i be mean independent of the remaining interaction effects and direct effects: $\text{E}(\beta_i|\beta_{-i}, \gamma_{-i}) = \text{E}(\beta_i)$. Then:

(i) If $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji} > 0$ with probability one, then $\beta_i^{\text{IV}}(z_i) > \text{E}(\beta_i)$.

(ii) If $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji} < 0$ with probability one, then $\beta_i^{\text{IV}}(z_i) < \text{E}(\beta_i)$.

Proposition 4 shows that the relationship between $\beta_i^{\text{IV}}(z_i)$ and $\text{E}(\beta_i)$ is governed by the sign of $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$, which measures how i's outcome Y_i affects the average peer outcome $\bar{Y}_{-i}$ in equilibrium. To interpret this quantity, recall that under Assumption II, the term ψ_{ji} shares the same sign as β_j. Hence, if all interaction effects share a common sign ($\text{sgn}(\beta_i) = \text{sgn}(\beta_j)$ for $i \neq j$), the IV estimand recovers an upper bound on the magnitude of $\text{E}(\beta_i)$. By contrast, in mixed-sign environments where i's interaction effect has the opposite sign from those of her peers (e.g., $\beta_i < 0$ and $\beta_j > 0$ for $j \neq i$), IV recovers a lower bound on the magnitude of $\text{E}(\beta_i)$. We next illustrate these arguments under the two- and three-agent special cases of the model:

Two-Agent Groups ($N = 2$). With two agents, statements (i) and (ii) in Proposition 4 become:

(i) If $\beta_j > 0$ with probability one, then $\beta_i^{\text{IV}}(z_i) > \text{E}(\beta_i)$.

(ii) If $\beta_j < 0$ with probability one, then $\beta_i^{\text{IV}}(z_i) < \text{E}(\beta_i)$.

Hence, the relationship between $\beta_i^{\text{IV}}(z_i)$ and $\text{E}(\beta_i)$ depends on the sign of agent j's interaction effect. In peer effects models where students prefer to conform ($\beta_j \geq 0$), IV recovers an upper bound on the average peer effect. In household labor-supply models where members do not place a negative weight on leisure ($\mu_j \geq 0$), IV recovers an upper bound on the magnitude of the average added earner effect. In oligopoly models where demand is downward-sloping ($b > 0$), IV recovers an upper bound on the magnitude of the average conduct parameter.

In each of these examples, we find that IV recovers an upper bound on the magnitude of $\mathrm{E}(\beta_i)$. This conclusion is not universal: when interaction effects have opposite signs in each group, IV instead recovers a lower bound on the magnitude of the average interaction effect.

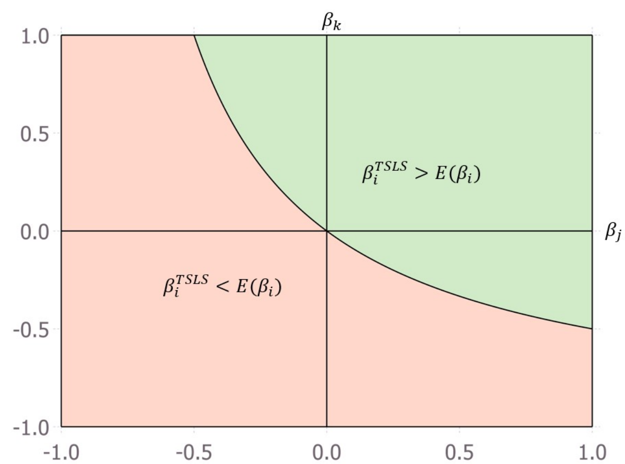
Three-Agent Groups ($N=3$). With three agents, statements (i) and (ii) in Proposition 4 are:

(i) If $\beta_j + \beta_k + \beta_j\beta_k > 0$ with probability one, then $\beta_i^{\mathrm{IV}}(z_i) > \mathrm{E}(\beta_i)$.

(ii) If $\beta_j + \beta_k + \beta_j\beta_k < 0$ with probability one, then $\beta_i^{\mathrm{IV}}(z_i) < \mathrm{E}(\beta_i)$.

Figure 2 plots the regions of (β_j, β_k) for which $\beta_j + \beta_k + \beta_j\beta_k > 0$, distinguishing cases when $\beta_i^{\mathrm{IV}}(z_i)$ lies above or below $\mathrm{E}(\beta_i)$. If β_j and β_k have the same sign, the ordering is immediate: two conforming peers imply $\beta_i^{\mathrm{IV}}(z_i) > \mathrm{E}(\beta_i)$; two differentiating peers imply $\beta_i^{\mathrm{IV}}(z_i) < \mathrm{E}(\beta_i)$. With mixed-sign interactions, the ordering depends on the relative strength of the interaction effects: one peer reinforces agent i's outcome while the other offsets it, and $\beta_i^{\mathrm{IV}}(z_i)$ lies above $\mathrm{E}(\beta_i)$ only when the reinforcing peer response dominates. Standard peer effects applications with uniformly positive interactions fall in the upper-right quadrant of Figure 2, while the household labor-supply and oligopoly examples fall in the lower-left quadrant. In both cases, the IV estimand provides an upper bound on the magnitude of the average interaction effect.

Figure 2: Upper-Bound Region for IV in Three-Agent Groups



Notes. This figure depicts regions of (β_j, β_k) for which IV estimands overstate the average interaction effect.

3.3.2 Using OLS to Test for Endogenous Interactions and Multipliers

We now show how OLS estimands can be used to test for endogenous interactions and social multipliers under heterogeneous effects. As a first step in deriving these tests, we establish a set of relationships linking our economic quantities of interest to the distribution of reduced-form effects. As OLS regressions recover averages of reduced-form effects, these relationships allow us to test hypotheses about the distribution of interaction effects and social multipliers.

Proposition 5. Suppose $\gamma_i > 0$ for each i. Under Assumptions I and II, the following hold:

(i) $M_{(i)} - 1$ has the same sign as $\Delta\bar{Y}_{-i}/\Delta Z_i$ for all i.

(ii) $M - 1$ has the same sign as $\sum_{i=1}^{N} \Delta\bar{Y}_{-i}/\Delta Z_i$.

(iii) For all $i \neq j$, the spillover effect $\Delta Y_i/\Delta Z_j$ has the same sign as β_i.

(iv) For any distinct i, j, k, the difference $\Delta Y_j/\Delta Z_i - \Delta Y_k/\Delta Z_i$ has the same sign as $\beta_j - \beta_k$.

(v) For all i, $\Delta Y_i/\Delta Z_i > 0$ and $\Delta\bar{Y}/\Delta Z_i > 0$.

Testing for Social Multipliers. We begin by showing how to use OLS estimands to learn about agent-specific social multipliers $M_{(i)}$ and aggregate multipliers M, defined in Table 1. In particular, we are interested in testing whether these multiplier effects lie above or below one: values above one indicate that spillovers amplify the effect of individual-level shocks on group-level outcomes, whereas values below one indicate that spillovers dampen these effects.

To draw inference on this question, we consider the reduced-form effect $\Delta\bar{Y}_{-i}/\Delta Z_i$, which measures how agent i's shifter Z_i affects the average peer outcome $\bar{Y}_{-i}$. By Proposition 5(i)-(ii), the sign of this effect governs whether the agent-specific multiplier $M_{(i)}$ is above or below one, and the sign of $\sum_{i=1}^{N} \Delta\bar{Y}_{-i}/\Delta Z_i$ plays an analogous role for the aggregate multiplier M.

Although these reduced-form effects cannot be recovered group by group, OLS regressions identify their averages $\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_i)$ for every i. OLS can therefore be used to test whether spillovers amplify or offset shocks for a measurable subset of groups. In particular, rejecting $H_0 : \mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_i) \leq 0$ implies $\mathrm{P}(M_{(i)} > 1) > 0$, while rejecting $H_0 : \sum_{i=1}^{N} \mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_i) \leq 0$ implies $\mathrm{P}(M > 1) > 0$. Analogous one-sided tests with the inequalities reversed offer insight into whether spillovers dampen, rather than amplify, the aggregate effect of individual shocks.

Testing for Interaction Effects. We next show how to test for the presence of endogenous interactions, as well as to learn about the signs of interaction effects, using OLS regressions. To do so, we draw on Proposition 5(iii). This result states that, whenever the direct effect γ_j has a known sign, the sign of the interaction effect β_i is revealed by the individual spillover effect $\Delta Y_i/\Delta Z_j$. Using this relationship, we can test for the presence of endogenous interactions by evaluating the null hypothesis $H_0 : \mathrm{E}(\Delta Y_i/\Delta Z_j) = 0$. Rejecting this null indicates that β_i is nonzero with positive probability. One-sided tests further provide evidence on the sign of β_i: under a monotonicity assumption $\mathrm{P}(\gamma_j \geq 0) = 1$, rejecting $H_0 : \mathrm{E}(\Delta Y_i/\Delta Z_j) \leq 0$ implies $\Pr(\beta_i > 0) > 0$; meanwhile, rejecting $H_0 : \mathrm{E}(\Delta Y_i/\Delta Z_j) \geq 0$ implies $\Pr(\beta_i < 0) > 0$.

The tests above rely on averages of spillover effects, $\mathrm{E}(\Delta Y_i/\Delta Z_j)$, which are identified from OLS regressions of Y_i on the vector of shifters Z. In some applications, however, these regressions may be impractical to run when the number of shifters is large. A natural dimension reduction is to replace the shifters Z_j with their average, but Section 3.2.1 shows that this aggregation is generally invalid under heterogeneity, as it imposes a common coefficient on shifters that have heterogeneous reduced form effects. We next show that, even though such aggregated regressions are misspecified for recovering average spillover effects, they can still be used to test for the presence of endogenous interactions. Consider the following result:

Lemma 7. For any agent i, if $\beta^{\mathrm{OLS}}_{\bar{Z}_{-i}}(Y_i; (1, Z_i, \bar{Z}_{-i})') \neq 0$, then $\beta_i \neq 0$ with positive probability.

This lemma shows that OLS regressions of outcomes on own and peer-average shifters remain informative about the presence of endogenous interactions under heterogeneity. Note that, as the coefficient on $\bar{Z}_{-i}$ combines heterogeneous spillover effects with nontrivial weights, its sign does not easily tell us whether the underlying interaction effects are positive or negative.

Assessing the Strength of Interaction Effects. We next show how OLS can be used to compare the strength of interaction effects among agents in a group. For any two agents j and k, we are interested in testing whether $\beta_j > \beta_k$ with positive probability. For example, in a peer effects model, we may want to evaluate whether male or female students experience greater social pressure. In a household labor-supply model, we may want to evaluate whether primary or secondary earners exhibit larger added-earner effects. In an oligopoly model, we may want to evaluate whether different types of firms possess different conduct parameters.

To conduct this test, we draw on Proposition 5(iv). This result states that, for any given agent $i \notin \{j, k\}$ for which $\gamma_i > 0$, the difference between j and k's interaction effects, $\beta_j - \beta_k$, has the same sign as the difference in the individual spillover effects, $\Delta Y_j/\Delta Z_i - \Delta Y_k/\Delta Z_i$. Therefore, under the monotonicity condition, $\mathrm{P}(\gamma_i \geq 0) = 1$, we can evaluate whether $\beta_j > \beta_k$ with positive probability by testing the null hypothesis $H_0 : \mathrm{E}(\Delta Y_j/\Delta Z_i) \leq \mathrm{E}(\Delta Y_k/\Delta Z_i)$.

Testing for Bounded Spillovers. Finally, OLS regressions can be used to test implications of Assumption II, which requires $\mathrm{P}(1 - N < \beta_i < 1) = 1$ for every agent i. Proposition 5(v) shows that, under this condition, the total individual effect $\Delta Y_i/\Delta Z_i$ and the total effect on the average $\Delta \bar{Y}/\Delta Z_i$ have the same sign as the direct effect γ_i. Thus, under the monotonicity condition $\Pr(\gamma_i \geq 0) = 1$, Assumption II implies $\mathrm{E}(\Delta Y_i/\Delta Z_i) \geq 0$ and $\mathrm{E}(\Delta \bar{Y}/\Delta Z_i) \geq 0$. We can therefore test Assumption II by testing the one-sided nulls $H_0 : \mathrm{E}(\Delta Y_i/\Delta Z_i) \geq 0$ and $H_0 : \mathrm{E}(\Delta \bar{Y}/\Delta Z_i) \geq 0$. Rejecting these nulls implies that spillovers are unbounded, which in most applications of the linear-in-means model suggests that the model may be misspecified.

4 Empirical Applications

We now apply our results to data from two empirical settings: peer effects in Kenyan primary schools, following Duflo et al. (2011), and strategic pricing decisions of cocoa traders in Sierra Leone, following Casaburi & Reed (2022). Both studies employ a linear-in-means model with homogeneous interaction effects. In each case, the model is over-identified, as individual-level shifters generate multiple exclusion restrictions within a group. We exploit this property to test the restriction of homogeneous interaction effects, finding that this restriction is rejected in both applications. We then reinterpret the estimates through the lens of heterogeneous interaction effects, drawing inference about endogenous interactions and social multipliers.

4.1 Classroom Peer Effects in Kenya

Our first application follows Duflo et al. (2011), who study the role of peer effects and ability tracking in Kenyan primary schools. The experiment included 121 schools, each divided into two classrooms. In treatment schools, students were assigned to classrooms based on ability, as measured by a baseline test score; in control schools, students were randomly assigned to classrooms. We follow Duflo et al. (2011) by restricting the sample to control schools when analyzing peer effects. This sample contains 2,849 students across 61 schools, each split into two rooms. After removing missing data, the sample retains 2,190 students over 48 schools.

To measure peer effects in classrooms, Duflo et al. (2011) consider the following model:

$$Y_i = \beta \bar{Y}_{-i} + Z_i' \gamma + \nu_s + \varepsilon_i, \tag{19}$$

where Y_i is the endline test score of a student i, $\bar{Y}_{-i}$ is the average endline test score of i's classmates, Z_i is a vector of controls that includes i's own baseline score, and ν_s is a school fixed effect. The authors use the average baseline score of i's classmates, $\bar{Z}_{-i}$, as an excluded instrument for $\bar{Y}_{-i}$. The outcome variables are math, reading, and total endline test scores.[15]

4.1.1 OLS/IV Estimates and Over-identification Tests

Table 3 presents results from our implementation of linear peer effects estimators. Columns 1-3 of Panel A show OLS estimates from regressing Y_i on Z_i and $\bar{Z}_{-i}$ with school fixed effects, the same specification in Duflo et al. (2011). Under homogeneous effects, these regressions recover the spillover effects, $\Delta Y_i/\Delta Z_j$, of any peer j's baseline score on a student i's endline score and the total individual effect, $\Delta Y_i/\Delta Z_i$, of i's baseline score on her own endline score.

Columns 4-6 of Panel A show estimates from OLS regressions of $\bar{Y}_{-i}$ on Z_i and $\bar{Z}_{-i}$ with school fixed effects. By Lemma 1, Assumptions A.1 and A.4 imply that the coefficient on Z_i in these regressions equals the coefficient on $\bar{Z}_{-i}$ in Columns 1-3. However, we find strong evidence against this equality, indicating that at least one of A.1 and A.4 fails. This rejection is consistent with within-classroom heterogeneity in peer effects β_i, direct effects γ_i, or both.

Columns 1-3 of Panel B report estimates for the main IV specification used in the original paper. Under homogeneous effects, these IV estimates recover the (homogeneous) peer effect β. Columns 4-6 of Table 3, Panel B, report estimates from alternative IV specifications that use multiple excluded instruments. In addition to $\bar{Z}_{-i}$, we construct four more instrumental

[15]Equation (19) corresponds to equation (E4) in the original paper, with notation adjusted to match ours.

variables: (1) the minimum baseline score of peers, (2) the maximum baseline score of peers, (3) the average baseline score of female peers, and (4) the average baseline score of male peers.

By Lemma 2, Assumption A.2 implies that any admissible combination of the excluded instruments should yield the same IV estimand. If Assumption A.2 fails, so that peer effects vary across classrooms, the IV estimand may instead depend on the choice of instruments. We therefore test the overidentifying restrictions implied by Assumption A.2 by conducting a Sargan-Hansen test using all five excluded instruments. This test assesses the overidentifying restrictions for any linear combination of excluded instruments.[16] We find that the test is rejected at the 5 percent level, indicating that peer effects are heterogeneous across classrooms.

Table 3: Classroom Peer Effects—Primary Schools in Kenya

	Own Endline Score			Peers' Mean Endline Score		
	Total (1)	Math (2)	Literature (3)	Total (4)	Math (5)	Literature (6)
Panel A. Reduced Form						
Own Baseline Score	0.507***	0.496***	0.413***	0.007**	0.006*	0.007**
	(0.026)	(0.022)	(0.030)	(0.003)	(0.003)	(0.003)
Peers' Mean Baseline Score	0.345**	0.324**	0.291**	0.788***	0.697***	0.704***
	(0.150)	(0.160)	(0.131)	(0.157)	(0.174)	(0.134)
Observations	2,188	2,188	2,189	2,188	2,188	2,189
	One Instrument Spec.			Multiple Instrument Spec.		
	Total	Math	Literature	Total	Math	Literature
Panel B. Instrumental Variables						
Peers' Mean Endline Score	0.444***	0.469***	0.422***	0.424***	0.488***	0.487***
	(0.117)	(0.124)	(0.120)	(0.094)	(0.103)	(0.117)
First-Stage F-Stat	371.8	371.6	1970	293.4	463.4	590.9
Sargan-Hansen Test[a]				15.12	12.53	12.76
				(0.004)	(0.014)	(0.013)
Observations	2,188	2,188	2,189	2,188	2,188	2,188

Notes. Data comes from Duflo et al. (2011). Following the authors' specifications, we include school fixed effects and controls for gender, age, and being assigned to the contract teacher. Columns (1)-(3) in Panel B use peers' mean baseline score as an excluded instrument. Columns (4)-(6) in Panel B use as excluded instruments: peers' mean baseline score, peers' minimum and maximum baseline scores, and mean baseline scores of male and female peers. Standard errors clustered at the school level.

[a]We report the Sargan-Hansen χ^2_4 test statistic with the corresponding p-value in parentheses below.
*p<0.1; **p<0.05; ***p<0.01.

[16]As usual, we maintain the standard instrument exogeneity condition in our empirical analysis, so violations of overidentifying restrictions can be interpreted as evidence against homogeneous interaction effects.

4.1.2 Reanalysis under Heterogeneous Interaction Effects

Motivated by these findings, we re-analyze the estimates in Table 3 under the linear-in-means model with heterogeneous interaction effects. Our analysis leverages two observations about the empirical setting. First, Assumption PM is likely to hold because students' baseline test scores are expected to have a nonnegative impact on their endline scores, making it plausible that $\mathrm{P}(\gamma_i \geq 0) = 1$ for all i. Second, Assumption NNW is satisfied, because the experimental design ensures that baseline scores $\{Z_j\}_{j=1}^N$ are uncorrelated after conditioning on the school.

Learning from OLS Estimates. Consider the OLS estimates in Columns 1-3 of Table 3, Panel A. Under heterogeneous interaction effects, the coefficient on Z_i in these regressions recovers the average total individual effect $\mathrm{E}(\Delta Y_i/\Delta Z_i)$. This quantity provides an aggregate summary measure of how students' baseline test scores affect their own endline test scores, after accounting for the role of peer effects. We estimate that, on average, scoring one point higher on a baseline test leads students to score about 0.5 points higher on the endline test.

In these OLS regressions, the coefficient on peers' mean baseline score $\bar{Z}_{-i}$ is found to be positive and statistically significant. By Lemma 7, this result allows us to conclude that peer effects are present in at least some share of classrooms. Note that, under heterogeneous effects, this estimate does not allow us to draw inference about the signs of the peer effects.[17]

Learning from IV Estimates. Consider the IV estimates in Table 3, Panel B. Proposition 3 shows that, under Assumptions PM and NNW, these estimates recover positively weighted averages of peer effects, with higher weight placed on classrooms where peers' average endline test scores, $\bar{Y}_{-i}$, are more responsive to their baseline test scores Z_{-i}. We estimate this weighted average peer effect to be about 0.45, and we find it to be statistically significant.

If we assume that all peer effects are nonnegative, such that $\mathrm{P}(\beta_i \geq 0) = 1$ for all i, then Proposition 4 implies that these IV estimates recover upper bounds on the average peer effect $\mathrm{E}(\beta_i)$. We can thus conclude that a one-point increase in peers' average test scores would not raise a student's own score by more than about 0.45 points on average. This upper bound is large, suggesting that peer effects could have a substantial impact on student achievement.

Testing for Social Multipliers. In Columns 4-6 of Table 3, Panel A, we estimate a statistically significant, positive coefficient on a student's own baseline test score Z_i, corresponding to the average equilibrium effect $\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_i)$.[18] By Proposition 5, this result tells us that social multipliers $M_{(i)}$ and M are greater than one in some positive share of classrooms. Thus, peer effects amplify the impacts of student-specific shocks on overall classroom achievement.

[17] In this application, it is infeasible to regress the outcomes Y on the entire vector $\tilde{Z} = (1, Z')'$ as it requires labeling each student i in a way that is consistent across classrooms. This task is more straightforward in some applications, for example, when studying labor supply in two-person households where there is always one primary earner. Yet, it is more difficult when the number and composition of agents in a group varies.

[18] This interpretation assumes that students' baseline scores $\{Z_j\}_{j=1}^N$ are uncorrelated with one another, which is implied by the experimental design, as students are randomly assigned to classrooms within a school.

4.2 Strategic Purchasing Decisions in Sierra Leone

Our second application builds on the analysis of Casaburi & Reed (2022), who study the strategic purchasing decisions of traders who buy cocoa from farmers in Sierra Leone. During an experiment conducted from October to December 2011, half of the 80 traders in the sample were randomly assigned a subsidy of 150 leones per pound of cocoa sold at village markets. Data on prices and quantities from the transactions were subsequently collected for analysis.

Casaburi & Reed (2022) write down a model of oligopsonistic competition among traders. Each market consists of N traders and a unit measure of homogeneous producers. The price P_i that trader i pays to producers is given by the inverse supply $P_i = \lambda + \kappa Q_i + \theta \sum_{j \neq i} Q_j$, which the authors microfound by assuming there is a representative producer with love for variety.[19] A trader's profit function is $\Pi_i = Q_i(v + sZ_i - P_i)$, where v is the (net) resale price of cocoa and Z_i indicates whether a given trader is randomly assigned a subsidy valued at s.

In equilibrium, each trader chooses a quantity Q_i to maximize profit, while accounting for the purchasing decisions $\{Q_j\}_{j \neq i}$ of other traders. The profit-maximizing quantities reduce to a linear-in-means model with constant effects, where the interaction effect β is $\theta(N-1)/2\kappa$:

$$\begin{aligned} Q_i &= \frac{v-\lambda}{2\kappa} - \frac{\theta}{2\kappa} \sum_{j \neq i} Q_j + \frac{s}{2\kappa} Z_i \\ &= \underbrace{\alpha}_{(v-\lambda)/2\kappa} + \underbrace{\frac{\beta}{N-1}}_{-\theta/2\kappa} \sum_{j \neq i} Q_j + \underbrace{\gamma}_{s/2\kappa} Z_i, \quad \text{for } i \in \{1, \dots, N\}. \end{aligned} \tag{20}$$

In this model, $\theta/2\kappa$ is a conduct parameter, measuring how a trader i's purchases depend on those of the other traders. Under constant effects, this parameter is identified from linear IV estimation, where $\sum_{j \neq i} Q_j$ is instrumented by the treatment statuses of i's rivals, $\{Z_j\}_{j \neq i}$.[20]

4.2.1 OLS/IV Estimates and Over-identification Tests

Table 4 presents our implementation of linear peer effects estimators. The first two columns of Panel A report OLS estimates from regressing a trader i's outcome Q_i on her own treatment status Z_i and the number of treated rivals $\sum_{j \neq i} Z_j$, with and without controls. With constant effects, this regression recovers the spillover effect $\Delta Y_i / \Delta Z_j$ of a rival j's subsidy on a trader i's outcome and the total individual effect $\Delta Y_i / \Delta Z_i$ of a trader's subsidy on her own outcome.

[19] Following footnote 6 in Casaburi & Reed (2022), a producer's profit is: $V(P, Q) = Q_0 + \sum_{i=1}^{N} P_i Q_i - C(Q)$, where $C(Q) = \lambda \sum_{i=1}^{N} Q_i + \frac{1}{2}\kappa \sum_{i=1}^{N} Q_i^2 + \theta \sum_{j \neq i} Q_i Q_j$ is the cost of production and Q_0 denotes unsold output.

[20] Casaburi & Reed (2022) do not estimate this IV specification because their analysis does not explicitly assign traders to markets. Instead, they impose additional structure to estimate market size N without ever defining the set of traders in each market. Our analysis requires such an assignment. We define a market as the interaction between a week and a chiefdom, which is a small administrative unit in Sierra Leone. In the data, we find that 90% of traders operate in a single chiefdom in a given week and more than 98% of traders make over half of their weekly sales in the same chiefdom. These observations support our market definition.

The final two columns of Panel A report estimates from regressing $\sum_{j \neq i} Q_j$ on Z_i and $\sum_{j \neq i} Z_j$, with and without controls. By Lemma 1, Assumptions A.1 and A.4 imply that the coefficient on $\bar{Z}_{-i}$ in a regression of Y_i on Z_i and $\bar{Z}_{-i}$ should equal the coefficient on Z_i in a regression of $\bar{Y}_{-i}$ on Z_i and $\bar{Z}_{-i}$. We are unable to reject this testable implication in the data.

The first two columns of Panel B present estimates from IV regressions of Q_i on $\sum_{j \neq i} Q_j$, where the number of treated competitors $\sum_{j \neq i} Z_j$ is the excluded instrument. In the classical linear-in-means model, this regression recovers the conduct parameter $-\theta/2\kappa$. The last two columns of Table 4, Panel B, report estimates from alternate IV specifications using multiple instruments. In addition to $\sum_{j \neq i} Z_j$, we introduce three extra instruments: (1) number of treated competitors who have access to a storage facility, (2) number of treated competitors older than the median age (37), and (3) number of treated competitors with baseline sales above the median (300 lbs of cocoa). Each of these instruments is valid by the same identification arguments used in the original paper. We use these over-identified regressions to test whether all traders share a common conduct parameter. We then conduct a Sargan–Hansen test for over-identifying restrictions using all four excluded instruments. From this exercise, we find strong evidence to reject the constant effects assumption. This finding suggests that different traders respond strategically in different ways to their rivals' purchasing decisions.

4.2.2 Reanalysis under Heterogeneous Interaction Effects

Motivated by these findings, we reanalyze the estimates in Table 4 under the linear-in-means model with heterogeneous effects. To do so, we first make two observations about the setting. First, Assumption PM is likely to hold since the coefficient γ_i is proportional to the subsidy s, which is homogeneous within and across markets. Second, Assumption NNW is satisfied because the experimental design ensures that treatments $\{Z_j\}_{j=1}^{N}$ are mutually uncorrelated.

Learning from OLS Estimates. Consider the OLS estimates in Columns 1-2 of Table 4, Panel A. Under heterogeneity, the coefficient on Z_i recovers $\mathrm{E}(\Delta Q_i/\Delta Z_i)$, which represents the average effect of receiving a subsidy on a trader's own purchases in equilibrium. We find that, on average, the subsidy induces a trader to purchase about 400 more pounds of cocoa.

In these OLS regressions, the coefficient on the number of treated competitors $\sum_{j \neq i} Z_j$ is estimated to be negative and statistically significant. By Lemma 7, this finding suggests that the conduct parameters $\theta_i/2\kappa_i$ are nonzero with positive probability. Therefore, at least some markets are imperfectly competitive, with traders internalizing their rivals' decisions.

Learning from IV Estimates. Consider the IV estimates in Table 4, Panel B. After including controls, we estimate a significant, negative IV estimand of -0.02. Under heterogeneous interaction effects, this estimand corresponds to a weighted average of conduct parameters among traders, with larger weights placed on traders whose competitors' purchases are more responsive to subsidies. As Assumptions PM and NNW are both likely to hold in this setting, Proposition 3 supports interpreting the estimand as a positively weighted average. The nega-

Table 4: Strategic Interactions—Cocoa Traders in Sierra Leone

	Trader Quantity		Competitors' Total Quantity	
	(1)	(2)	(3)	(4)
Panel A. Reduced Form				
Treatment Trader	416.663***	454.895***	-166.995	-61.516
	(45.733)	(49.594)	(248.156)	(267.626)
Number of Treated Competitors	-10.733***	-7.423**	507.685***	522.394***
	(2.975)	(3.697)	(16.141)	(19.948)
Observations	610	602	610	602
Trader Controls		X		X
	One Instrument Spec.		**Multiple Instrument Spec.**	
	(1)	(2)	(3)	(4)
Panel B. Instrumental Variables				
Competitors' Total Quantity	-0.007	-0.020***	-0.004	-0.018***
	(0.006)	(0.007)	(0.006)	(0.007)
First-Stage F-Stat	23.06	14.15	22.90	14.09
Sargan-Hansen Test[a]			9.82	12.35
			(0.02)	(0.006)
Observations	610	602	610	602
Trader Controls		X		X

Notes. Data comes from Casaburi & Reed (2022). Following the original paper, we include week fixed effects. Trader controls are: baseline pounds of cocoa sold, number of villages where trader operates, baseline share of suppliers receiving credit from trader, age, years working with wholesaler, ownership of a cement or tile floor, mobile phone, and access to a storage facility. Sample sizes differ between (1) and (2) due to missing data about trader controls.

[a]We report a Sargan-Hansen χ^2_3 test statistic with a corresponding p-value in parentheses.
*p<0.1; **p<0.05; ***p<0.01.

tive estimate thus provides insight into the extent of strategic substitutability among traders.

Because traders' conduct parameters $\theta_i/2\kappa_i$ are positive by construction, the IV estimate recovers an upper bound on the magnitude of the average conduct parameter $\mathrm{E}(\theta_i/2\kappa_i)$. We thus conclude that, on average, raising a rival trader's cocoa purchases by 1 pound does not reduce a trader's own purchases by more than 0.02 pounds. This upper bound is small, which tells us that, while strategic interactions exist, they are nonetheless limited in this context.

Testing for Social Multipliers. Consider the OLS estimates in Columns 3-4 of Table 4, Panel A. The coefficient on Z_i recovers $\mathrm{E}\left(\Delta\big(\sum_{j\neq i} Q_j\big)/\Delta Z_i\right)$, measuring the average effect of a trader i's treatment status on total quantity purchased by rival traders in equilibrium.[21]

[21]As in the first application, this interpretation requires that $\{Z_j\}_{j=1}^N$ are uncorrelated with one another.

We estimate this coefficient to be small and statistically insignificant, providing no evidence of social multiplier effects in this setting. Thus, we conclude that strategic interactions have little to no material impact on how changes in traders' demands affect overall market output.

5 Conclusion

This paper has developed a set of tools for empirically analyzing linear-in-means models with heterogeneous interaction effects. The classical linear-in-means model, widely applied in the peer effects literature, imposes that endogenous interaction effects are identical within and across peer groups. This homogeneity restriction delivers sharp testable implications that we showed can be readily examined in data. We relaxed these restrictions to allow interaction effects to differ in both sign and magnitude, within and across groups. We showed that this extension allows the model to accommodate richer forms of economic behavior, including peer effects with both conformity and differentiation, joint labor-supply decisions in heterogeneous households, and strategic interactions among firms with different cost structures and conduct.

We then characterized the identifying content of commonly used OLS and IV estimands for linear-in-means models once the standard homogeneity restriction is relaxed. We showed that these estimands generally do not point identify the structural parameters. Nevertheless, they remain informative about relevant reduced form quantities, social multipliers, and the distribution of interaction effects. Using these insights, we explained how OLS and IV estimands can be used to test for endogenous interactions and social multipliers, and to bound unweighted averages of interaction effects. We applied these results to two empirical settings: peer effects in Kenyan primary schools and strategic pricing of cocoa traders in Sierra Leone. In both cases, we rejected homogeneous interaction effects; yet, under the heterogeneous effects framework, we could still draw meaningful conclusions about endogenous interactions and social multipliers. Our findings highlighted that commonly used linear peer effects estimands retain empirical value under heterogeneity, but that interpreting them requires care.

This condition is ensured by the experimental protocols, as a trader's treatment status is randomly assigned.

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Supplemental Appendix

A.1. Derivation of Reduced Form Quantities

Proof of Proposition 1

Consider a group with N agents, where agents' outcomes are defined by the system (1).[22] To prove Proposition 1, we begin by defining the reduced form system using matrix notation.

$$Y = \det(I - \mathbf{B})^{-1}\mathbf{A}[\alpha + \mathrm{diag}(\gamma)Z],$$

where $\mathbf{A} = \mathrm{adj}(I - \mathbf{B})$ is the adjugate of $I - \mathbf{B}$, and $\det(I - \mathbf{B})$ is the determinant of $I - \mathbf{B}$. By definition, $\mathbf{A}$ is equal to the transpose of the matrix of cofactors of $I - \mathbf{B}$. In particular, the individual entries $\{A_{ij}\}_{i,j}$ of the matrix $\mathbf{A}$ are defined so that:

$$A_{ij} = (-1)^{i+j} \times \det([I - \mathbf{B}]_{-j,-i}),$$

where $[I - \mathbf{B}]_{-j,-i}$ is a submatrix formed by removing the jth row and ith column of $I - \mathbf{B}$.

We want to derive alternate expressions for $\{A_{ij}\}_{i,j}$ that are not in matrix form. To do so, we write $A_{ij} = (-1)^{i+j} \times \det\big(\mathbf{C}(i,j) - (N-1)^{-1}\beta_{-j}\mathbf{1}'_{(N-1)\times 1}\big)$, where $\mathbf{C}(i,j) \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix that is given by $\mathbf{C}(i,j) = I_{-j,-i}(\mathbf{1}_{(N-1)\times 1} + (N-1)^{-1}\beta_{-j})$. This matrix satisfies:

$$\det\big(\mathbf{C}(i,j)\big) = \mathbb{1}\{i = j\} \times \prod_{\ell \neq j}\Big(1 + \frac{\beta_\ell}{N-1}\Big)$$

$$\mathrm{adj}\big(\mathbf{C}(i,j)\big) = \begin{cases} \mathrm{diag}\left(\left\{\prod_{\ell \notin \{k,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\right\}_{k \neq j}\right) & \text{if } i = j \\ \left[(-1)^{i+j-1} \times \prod_{\ell \notin \{i,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\right][\mathrm{e}_j]_{-i}[\mathrm{e}_i]'_{-j} & \text{if } i \neq j \end{cases}$$

Then, by the matrix determinant lemma, the diagonal entries $\{A_{jj}\}_{j=1}^N$ of $\mathbf{A}$ are equal to:

$$\begin{aligned} A_{jj} &= \det\big(\mathbf{C}(j,j)\big) - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\,\mathrm{adj}\big(\mathbf{C}(j,j)\big)\beta_{-j} \\ &= \prod_{\ell \neq j}\Big(1 + \frac{\beta_\ell}{N-1}\Big) - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\,\mathrm{diag}\left(\left\{\prod_{\ell \notin \{k,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\right\}_{k \neq j}\right)\beta_{-j} \\ &= \prod_{\ell \neq j}\Big(1 + \frac{\beta_\ell}{N-1}\Big) - \sum_{k \neq j}\left[\frac{\beta_k}{N-1}\prod_{\ell \notin \{k,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\right] \end{aligned}$$

[22] To simplify the notation, we will omit group subscripts and treat Z_i as a one-dimensional variable.

Moreover, by the exact same reasoning, the off-diagonal entries $\{A_{ij}\}_{i\neq j}$ of $\mathbf{A}$ are equal to:

$$\begin{aligned}
A_{ij} &= (-1)^{i+j} \times \Big[\det\big(\mathbf{C}(i,j)\big) - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\operatorname{adj}\big(\mathbf{C}(i,j)\big)\beta_{-j}\Big] \\
&= (-1)^{i+j} \times \Big[0 - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\Big[(-1)^{i+j-1} \times \prod_{\ell\notin\{i,j\}}\Big(1+\frac{\beta_\ell}{N-1}\Big)\Big][\mathrm{e}_j]_{-i}[\mathrm{e}_i]'_{-j}\beta_{-j}\Big] \\
&= \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\prod_{\ell\notin\{i,j\}}\Big(1+\frac{\beta_\ell}{N-1}\Big)[\mathrm{e}_j]_{-i}[\mathrm{e}_i]'_{-j}\beta_{-j} \\
&= \frac{\beta_i}{N-1}\prod_{\ell\notin\{i,j\}}\Big(1+\frac{\beta_\ell}{N-1}\Big)
\end{aligned}$$

Now that we have derived these expressions for $\{A_{ij}\}_{i,j}$, our next step is to re-write the determinant of $I-\mathbf{B}$ so that it is not in matrix form. To do so, we take the following steps:

$$\begin{aligned}
\det(I-\mathbf{B}) &= \det\Big[I + \frac{1}{N-1}\operatorname{diag}(\beta) - \frac{1}{N-1}\beta\mathbf{1}'_{N\times 1}\Big] \\
&= \det\Big[I + \frac{1}{N-1}\operatorname{diag}(\beta)\Big]\bigg(1 - \frac{1}{N-1}\mathbf{1}'_{N\times 1}\Big[I + \frac{1}{N-1}\operatorname{diag}(\beta)\Big]^{-1}\beta\bigg)
\end{aligned}$$

For any agent $i \in \{1,\ldots,N\}$, this determinant can be reformulated as:

$$\begin{aligned}
\det(I-\mathbf{B}) &= \prod_{\ell=1}^{N}\Big(1+\frac{\beta_\ell}{N-1}\Big) \times \Big[1 - \sum_{j=1}^{N}\frac{\beta_j}{N-1}\Big(1+\frac{\beta_j}{N-1}\Big)^{-1}\Big] \\
&= \prod_{\ell\neq i}\Big(1+\frac{\beta_\ell}{N-1}\Big) \times \Big[1 - \Big(1+\frac{\beta_i}{N-1}\Big)\sum_{j\neq i}\frac{\beta_j}{N-1}\Big(1+\frac{\beta_j}{N-1}\Big)^{-1}\Big] \\
&= A_{ii} - \frac{\beta_i}{N-1}\sum_{j\neq i}\Big[\frac{\beta_j}{N-1}\prod_{\ell\notin\{i,j\}}\Big(1+\frac{\beta_\ell}{N-1}\Big)\Big]
\end{aligned}$$

By plugging in our expressions for $\{A_{ij}\}_{i,j}$ and $\det(I-\mathbf{B})$, we are now able to write down the ith reduced form equation for any agent $i \in \{1,\ldots,N\}$. This equation is given by:

$$\begin{aligned}
Y_i &= \frac{1}{\det(I-\mathbf{B})}\Big[A_{ii}(\alpha_i+\gamma_i Z_i) + \sum_{j\neq i}A_{ij}(\alpha_j+\gamma_j Z_j)\Big] \\
&= \alpha_i + \gamma_i Z_i + \frac{\sum_{j\neq i}\zeta_{ij} \times \Big[\frac{\beta_j}{N-1}(\alpha_i+\gamma_i Z_i) + (\alpha_j+\gamma_j Z_j)\Big]}{\det(I-\mathbf{B})}
\end{aligned}$$

where $\zeta_{ij} = \frac{\beta_i}{N-1}\prod_{\ell\notin\{i,j\}}\big(1+\frac{\beta_\ell}{N-1}\big)$. Next, for any $i \in \{1,\ldots,N\}$, consider the average outcome $\bar{Y}_{-i}$ among everyone excluding agent i. To derive an expression for $\bar{Y}_{-i}$, we write:

$$\begin{aligned}
\bar{Y}_{-i} &= \frac{1}{(N-1)\times\det(I-\mathbf{B})} \times (\mathbf{1}_{N\times 1} - \mathrm{e}_i)'\mathbf{A}[\alpha + \operatorname{diag}(\gamma)Z] \\
&= \frac{1}{(N-1)\times\det(I-\mathbf{B})} \times \sum_{j=1}^{N}\underbrace{\Big[\sum_{k\neq i}A_{kj}\Big]}_{c_{ij}}(\alpha_j+\gamma_j Z_j),
\end{aligned}$$

where the coefficient $c_{ii} = \sum_{k \neq i} A_{ki}$ is defined to be:

$$c_{ii} = \sum_{k \neq i} \left[\frac{\beta_k}{N-1} \prod_{\ell \notin \{k,i\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \right],$$

and where each of the coefficients $c_{ij} = \sum_{k \neq i} A_{kj}$, for $j \neq i$, is defined to be:

$$\begin{aligned}
c_{ij} &= A_{jj} + \sum_{k \notin \{i,j\}} A_{kj} \\
&= \prod_{\ell \neq j} \Big(1 + \frac{\beta_\ell}{N-1}\Big) - \sum_{k \neq j} \left[\frac{\beta_k}{N-1} \prod_{\ell \notin \{k,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \right] + \sum_{k \notin \{i,j\}} \left[\frac{\beta_k}{N-1} \prod_{\ell \notin \{k,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \right] \\
&= \prod_{\ell \neq j} \Big(1 + \frac{\beta_\ell}{N-1}\Big) - \frac{\beta_i}{N-1} \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \\
&= \Big(1 + \frac{\beta_i}{N-1} - \frac{\beta_i}{N-1}\Big) \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \\
&= \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big)
\end{aligned}$$

After plugging in these expressions for $\{c_{ij}\}_{j=1}^N$, we arrive at the following equation:

$$\bar{Y}_{-i} = \frac{\sum_{j \neq i} c_{ij} \times \left[\frac{\beta_j}{N-1}(\alpha_i + \gamma_i Z_i) + (\alpha_j + \gamma_j Z_j) \right]}{(N-1) \times \det(I - \mathbf{B})}$$

Through analogous steps, we can also derive an expression for the mean group outcome $\bar{Y}$:

$$\bar{Y} = \frac{\sum_{j=1}^N c_j \times (\alpha_j + \gamma_j Z_j)}{N \times \det(I - \mathbf{B})}, \quad \text{where} \quad c_j = \prod_{\ell \neq j} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \quad \text{for} \quad j \in \{1, \ldots, N\}$$

□

Necessary and Sufficient Conditions for a Unique Equilibrium

A unique equilibrium exists if and only if the determinant of $I - \mathbf{B}$ is nonzero. We write:

$$\begin{aligned}
\det(I - \mathbf{B}) &= \prod_{j=1}^N \Big(1 + \frac{\beta_j}{N-1}\Big) \times \left[1 - \sum_{i=1}^N \frac{\beta_i}{N-1} \Big(1 + \frac{\beta_i}{N-1}\Big)^{-1} \right] \\
&= \sum_{i=1}^N \left[\frac{1}{N} \prod_{j=1}^N \Big(1 + \frac{\beta_j}{N-1}\Big) - \frac{\beta_i}{N-1} \prod_{j \neq i} \Big(1 + \frac{\beta_j}{N-1}\Big) \right] \\
&= \Big(\frac{N-1}{N}\Big) \sum_{i=1}^N (1 - \beta_i) \prod_{j \neq i} (N - 1 + \beta_j)
\end{aligned}$$

So, for any $N \geq 2$, a unique equilibrium exists if and only if $\sum_{i=1}^N (1 - \beta_i) \prod_{j \neq i} (N - 1 + \beta_j) \neq 0$.

Derivation of Social Multipliers

We begin by deriving a closed-form expression for the agent-specific social multiplier $M_{(i)}$:

$$
\begin{aligned}
M_{(i)} = \frac{\sum_{j=1}^{N} \Delta Y_j / \Delta Z_i}{\Delta Y_i / \Delta Z_i} &= \frac{\frac{\gamma_i \nu_i}{\det(I-\mathbf{B})}}{\gamma_i + \frac{\beta_i \gamma_i \left(\frac{1}{N-1}\sum_{j \neq i} \beta_j \nu_{ij}\right)}{(N-1)\det(I-\mathbf{B})}} \\
&= \frac{\left(1+\frac{\beta_i}{N-1}\right)^{-1}}{1-\sum_{j=1}^{N}\frac{\beta_j}{N-1}\left(1+\frac{\beta_j}{N-1}\right)^{-1} + \frac{\beta_i}{N-1}\left(1+\frac{\beta_i}{N-1}\right)^{-1}\left(\sum_{j\neq i}\frac{\beta_j}{N-1}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\right)} \\
&= \frac{\left(1+\frac{\beta_i}{N-1}\right)^{-1}}{1-\frac{\beta_i}{N-1}\left(1+\frac{\beta_i}{N-1}\right)^{-1} + \left[\frac{\beta_i}{N-1}\left(1+\frac{\beta_i}{N-1}\right)^{-1}-1\right]\left(\sum_{j\neq i}\frac{\beta_j}{N-1}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\right)} \\
&= \frac{\left(1+\frac{\beta_i}{N-1}\right)^{-1}}{\left(1+\frac{\beta_i}{N-1}\right)^{-1}\left[1-\left(\sum_{j\neq i}\frac{\beta_j}{N-1}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\right)\right]} \\
&= \frac{1}{1-\left(\sum_{j\neq i}\frac{\beta_j}{N-1}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\right)}
\end{aligned}
$$

We next derive an analogous closed-form expression for the aggregate social multiplier M:

$$
\begin{aligned}
M = \frac{\sum_{i=1}^{N} \Delta \bar{Y} / \Delta Z_i}{\frac{1}{N}\sum_{i=1}^{N} \Delta Y_i / \Delta Z_i} &= \frac{\sum_{i=1}^{N} \frac{\gamma_i \nu_i}{N \times \det(I-\mathbf{B})}}{\frac{1}{N}\sum_{i=1}^{N} \frac{\gamma_i\left(\nu_i - \sum_{j\neq i}\frac{\beta_j}{N-1}\nu_{ij}\right)}{\det(I-\mathbf{B})}} \\
&= \frac{\frac{1}{N}\sum_{i=1}^{N}\gamma_i\nu_i}{\frac{1}{N}\sum_{i=1}^{N}\gamma_i\left(\nu_i - \sum_{j\neq i}\frac{\beta_j}{N-1}\nu_{ij}\right)} \\
&= \frac{\frac{1}{N}\sum_{i=1}^{N}\left(1+\frac{\beta_i}{N-1}\right)^{-1}\gamma_i}{\frac{1}{N}\sum_{i=1}^{N}\left(1+\frac{\beta_i}{N-1}\right)^{-1}\left(1-\sum_{j\neq i}\frac{\beta_j}{N-1}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\right)\gamma_i}.
\end{aligned}
$$

A.2. Proofs of Identification Results

This appendix subsection provides proofs of the main identification results in the paper. We omit the proof of Lemma 1, which follows directly from our derived closed-form expressions for the reduced-form effects reported in Table 1. We also omit the proofs of Proposition 2 and Lemma 2, since these results are direct applications of the classical identification arguments for linear OLS and IV estimands reviewed in textbooks dating back at least to Fisher (1966).

Proof of Lemma 3

Let Z_j and Z_k be binary variables so that (Z_j, Z_k) takes values in $\{(0,0),(0,1),(1,0),(1,1)\}$. Given this set of possible values, Assumption IAM specializes to four separate restrictions:

(1) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \leq 0\right) = 1$

(2) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

(3) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} + \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} + \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

(4) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} - \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} - \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

Under Assumption II, the reduced form effects $\Delta \bar{Y}_{-i}/\Delta Z_j$ and $\Delta \bar{Y}_{-i}/\Delta Z_k$ always share the same signs as γ_j and γ_k, respectively. Therefore, restrictions (1) and (2) are equivalent to:

(1') $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$

(2') $\mathrm{P}(\gamma_k \geq 0) = 1$ or $\mathrm{P}(\gamma_k \leq 0) = 1$

When combined with with (1) and (2), restrictions (3) and (4) simplify to a single condition:

(3') $\mathrm{P}(|\Delta \bar{Y}_{-i}/\Delta Z_j| \geq |\Delta \bar{Y}_{-i}/\Delta Z_k|) = 1$ or $\mathrm{P}(|\Delta \bar{Y}_{-i}/\Delta Z_j| \leq |\Delta \bar{Y}_{-i}/\Delta Z_k|) = 1$.

We can reinterpret this condition as a statement about the random coefficients by writing:

(3") $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \geq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$ or $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \leq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$

□

Proof of Lemma 4

Suppose Z_j and Z_k are continuous variables, and consider any two vectors (z_j, z_k) and (z_j', z_k') taken from the support of (Z_j, Z_k). The difference in $\bar{Y}_{-i}$ when evaluated at these vectors is:

$$\begin{aligned}\bar{Y}_{-i}(z_j, z_k) - \bar{Y}_{-i}(z_j', z_k') &= \left[\bar{Y}_{-i}(z_j, z_k) - \bar{Y}_{-i}(z_j', z_k)\right] + \left[\bar{Y}_{-i}(z_j', z_k) - \bar{Y}_{-i}(z_j', z_k')\right] \\ &= \frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \times (z_j - z_j') + \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \times (z_k - z_k')\end{aligned}$$

Assumption IAM requires that $\bar{Y}_{-i}^{g_1}(z_j, z_k) - \bar{Y}_{-i}^{g_1}(z_j', z_k')$ and $\bar{Y}_{-i}^{g_2}(z_j, z_k) - \bar{Y}_{-i}^{g_2}(z_j', z_k')$ share the same sign for any two groups g_1 and g_2. We show that the condition holds if and only if:

(1) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \leq 0\right) = 1$

(2) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

(3) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}/\Delta Z_j}{\Delta \bar{Y}_{-i}/\Delta Z_k} = a\right) = 1$ for some $a \in \mathbb{R}$

("$\Leftarrow$") Suppose Assumption IAM holds. Then, (1) and (2) are satisfied for the same reason that they hold in the binary case. To justify (3), let (z_j, z_k) be any vector that lies within the interior of the support of (Z_j, Z_k). For two groups g_1 and g_2, define the quantities:

$$z_j' = z_j - \left[\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_k} + \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_k}\right] \times \epsilon \quad \text{and} \quad z_k' = z_k + \left[\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_j} + \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_j}\right] \times \epsilon,$$

where $\epsilon > 0$ is chosen to be sufficiently small so that (z'_j, z'_k) lies inside the support of (Z_j, Z_k). In this case, the differences $\bar{Y}^{g_1}_{-i}(z_j, z_k) - \bar{Y}^{g_1}_{-i}(z'_j, z'_k)$ and $\bar{Y}^{g_2}_{-i}(z_j, z_k) - \bar{Y}^{g_2}_{-i}(z'_j, z'_k)$ are equal to:

$$\bar{Y}^{g_1}_{-i}(z_j, z_k) - \bar{Y}^{g_1}_{-i}(z'_j, z'_k) = \left(\frac{\Delta\bar{Y}^{g_1}_{-i}}{\Delta Z_j} \times \frac{\Delta\bar{Y}^{g_2}_{-i}}{\Delta Z_k}\right)\epsilon - \left(\frac{\Delta\bar{Y}^{g_1}_{-i}}{\Delta Z_k} \times \frac{\Delta\bar{Y}^{g_2}_{-i}}{\Delta Z_j}\right)\epsilon$$
$$\bar{Y}^{g_2}_{-i}(z_j, z_k) - \bar{Y}^{g_2}_{-i}(z'_j, z'_k) = \left(\frac{\Delta\bar{Y}^{g_1}_{-i}}{\Delta Z_k} \times \frac{\Delta\bar{Y}^{g_2}_{-i}}{\Delta Z_j}\right)\epsilon - \left(\frac{\Delta\bar{Y}^{g_1}_{-i}}{\Delta Z_j} \times \frac{\Delta\bar{Y}^{g_2}_{-i}}{\Delta Z_k}\right)\epsilon$$

Observe that the first equation is equal to the negative of the second equation. Thus, these differences can only share the same sign when they both equal zero. Specifically, we require:

$$\frac{\Delta\bar{Y}^{g_1}_{-i}}{\Delta Z_j} \times \frac{\Delta\bar{Y}^{g_2}_{-i}}{\Delta Z_k} = \frac{\Delta\bar{Y}^{g_1}_{-i}}{\Delta Z_k} \times \frac{\Delta\bar{Y}^{g_2}_{-i}}{\Delta Z_j} \quad \Longleftrightarrow \quad \frac{\Delta\bar{Y}^{g_1}_{-i}/\Delta Z_j}{\Delta\bar{Y}^{g_1}_{-i}/\Delta Z_k} = \frac{\Delta\bar{Y}^{g_2}_{-i}/\Delta Z_j}{\Delta\bar{Y}^{g_2}_{-i}/\Delta Z_k}$$

This equation holds for any two groups g_1 and g_2. So, $\mathrm{P}\left(\frac{\Delta\bar{Y}_{-i}/\Delta Z_j}{\Delta\bar{Y}_{-i}/\Delta Z_k} = a\right) = 1$ for some $a \in \mathbb{R}$.

("$\Rightarrow$") Suppose that conditions (1), (2), and (3) apply. Then, for some constant $a \in \mathbb{R}$:

$$\bar{Y}_{-i}(z_j, z_k) - \bar{Y}_{-i}(z'_j, z'_k) = \frac{\Delta\bar{Y}_{-i}}{\Delta Z_k} \times \left[a \times (z_j - z'_j) + (z_k - z'_k)\right],$$

where $\Delta\bar{Y}_{-i}/\Delta Z_k$ retains the same sign across groups. Thus, Assumption IAM must apply. Under Assumption II, we can reinterpret (1), (2), and (3) in terms of the random coefficients:

(1') $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$

(2') $\mathrm{P}(\gamma_k \geq 0) = 1$ or $\mathrm{P}(\gamma_k \leq 0) = 1$

(3') $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} = a \times \frac{\gamma_j}{\gamma_k}\right) = 1$ for some $a \in \mathbb{R}$

□

Proof of Lemma 5

For any $j \neq i$, consider any two vectors $(z_j, \{z_k\}_{k\notin\{i,j\}})$ and $(z'_j, \{z_k\}_{k\notin\{i,j\}})$ in the support of Z_{-i}. By Proposition 1, the difference between the values of $\bar{Y}_{-i}$ evaluated at these vectors is:

$$\bar{Y}_{-i}(z_j, \{z_k\}_{k\notin\{i,j\}}) - \bar{Y}_{-i}(z'_j, \{z_k\}_{k\notin\{i,j\}}) = \frac{\prod_{\ell\notin\{i,j\}}\left(1 + \frac{\beta_\ell}{N-1}\right) \times \gamma_j(z_j - z'_j)}{(N-1) \times \det(I - \mathbf{B})}$$

Under Assumption II, $\det(I - \mathbf{B}) > 0$ and $\prod_{\ell\notin\{i,j\}}\left(1 + \frac{\beta_\ell}{N-1}\right) > 0$ with probability one. Thus, $\bar{Y}_{-i}(z_j, \{z_k\}_{k\notin\{i,j\}}) - \bar{Y}_{-i}(z'_j, \{z_k\}_{k\notin\{i,j\}})$ retains a common sign across groups if and only if:

$$\mathrm{P}\big(\gamma_j(z_j - z'_j) \geq 0\big) = 1 \quad \text{or} \quad \mathrm{P}\big(\gamma_j(z_j - z'_j) \leq 0\big) = 1,$$

which occurs if and only if $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$. This condition applies for any j.

□

Proof of Lemma 6

This result immediately follows from the observation that, under Assumption II, the spillover effect $\Delta\bar{Y}_{-i}/\Delta Z_j$ of Z_j on $\bar{Y}_{-i}$ always shares the same sign as γ_j for every $j \in \{1, \dots, N\} \setminus i$. $\square$

Proof of Lemma 7

We prove the contrapositive. Under heterogeneous interaction effects, the outcomes $\{Y_i\}_{i=1}^N$ satisfy $Y_i = \pi_{i0} + \pi_{ii} Z_i + \sum_{j \neq i} \pi_{ij} Z_j$ in equilibrium, where $\pi_{ij} = \Delta Y_i/\Delta Z_j$. Using the reduced form formula derived in Proposition 1, the peer-shifter component of Y_i can be written as:

$$\sum_{j \neq i} \pi_{ij} \times Z_j = \beta_i \times \sum_{j \neq i} \frac{\gamma_j \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{N-1}\right)}{(N-1)\det(I - \mathbf{B})} \times Z_j$$

If $\mathrm{P}(\beta_i = 0) = 1$, then $\mathrm{P}(\pi_{ij} = 0) = 1$ for all $j \neq i$, so $Y_i = \pi_{i0} + \pi_{ii} Z_i$. Let $\tilde{Y}_i$ and $\tilde{\bar{Z}}_{-i}$ be the residuals from OLS regressions of Y_i and $\bar{Z}_{-i}$ on $(1, Z_i)'$. By the Frisch-Waugh-Lovell theorem:

$$\beta^{\mathrm{OLS}}_{\bar{Z}_{-i}}(Y_i; (1, Z_i, \bar{Z}_{-i})') = \frac{\mathrm{Cov}(\tilde{\bar{Z}}_{-i}, \tilde{Y}_i)}{\mathrm{Var}(\tilde{\bar{Z}}_{-i})}.$$

Under instrument independence, $(\pi_{i0}, \pi_{ii}) \perp Z$, so the linear projection of Y_i on $(1, Z_i)'$ equals $\mathrm{E}(\pi_{i0}) + \mathrm{E}(\pi_{ii}) Z_i$. Thus, $\tilde{Y}_i = \pi_{i0} - \mathrm{E}(\pi_{i0}) + [\pi_{ii} - \mathrm{E}(\pi_{ii})] Z_i$. Because $\tilde{\bar{Z}}_{-i}$ is residualized on $(1, Z_i)'$, it is uncorrelated with Z_i. Thus, $\mathrm{Cov}(\tilde{\bar{Z}}_{-i}, \tilde{Y}_i) = 0$, and so $\beta^{\mathrm{OLS}}_{\bar{Z}_{-i}}(Y_i; (1, Z_i, \bar{Z}_{-i})') = 0$. It follows that, if $\beta^{\mathrm{OLS}}_{\bar{Z}_{-i}}(Y_i; (1, Z_i, \bar{Z}_{-i})')$ is nonzero, then β_i is nonzero with positive probability. $\square$

Proof of Proposition 3

In this model, $\bar{Y}_{-i}$ is a linear function of Z. Therefore, we can write $\bar{Y}_{-i} = \pi_0 + \sum_{j=1}^N \pi_j Z_j$ for some parameters π_0 and $\{\pi_j\}_{j=1}^N$ that depend on the random coefficient vector $(\alpha, \beta, \gamma, \mathcal{N})$. Because the random coefficients are independent of Z, the conditional expectation of $\bar{Y}_{-i}$ given Z is equal to $\mathrm{E}(\bar{Y}_{-i}|Z) = \mathrm{E}(\pi_0) + \sum_{j=1}^N \mathrm{E}(\pi_j) Z_j$. Given these properties, we can write:

$$\begin{aligned}
\beta_i^{\mathrm{IV}}(z_i) = \frac{\mathrm{Cov}(Y_i, \mathbf{L}(\bar{Y}_{-i}|Z_{-i})|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathbf{L}(\bar{Y}_{-i}|Z_{-i})|Z_i = z_i)} &= \frac{\mathrm{Cov}(Y_i, \mathrm{E}(\bar{Y}_{-i}|Z)|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathrm{E}(\bar{Y}_{-i}|Z)|Z_i = z_i)} \\
&= \sum_{j \neq i} \mathrm{E}(\pi_j) \times \frac{\mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathrm{E}(\bar{Y}_{-i}|Z_{-i})|Z_i = z_i)} \\
&= \sum_{j \neq i} \mathrm{E}(\pi_j) \times \frac{\mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} \\
&= \sum_{j \neq i} \underbrace{\frac{\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)}}_{\omega_j} \times \frac{\mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)}
\end{aligned}$$

By construction, the weights $\{\omega_j\}_{j \neq i}$ sum to one. Furthermore, we prove the following claim.

Claim 1. Suppose that Assumption NNW holds. Then ω_j will be non-negative for all $j \neq i$.

Proof. For $j \neq i$, the weight ω_j is non-negative if and only if its numerator and denominator have the same sign. So, $\{\omega_j\}_{j \neq i}$ are non-negative if and only if $\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_i = z_i)$ has the same sign as $\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)$ for all $j \neq i$. Note that this statement is equivalent to the requirement that $\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_i = z_i)$ retains the same sign across all $j \neq i$. Therefore, for any $j, k \in \{1, \ldots, N\} \setminus i$, we rule out the case where:

$$
\begin{aligned}
0 &> \mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_i = z_i) \\
&= \mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)\,\mathrm{Cov}(Z_j, Z_k | Z_i = z_i) + \sum_{\ell \notin \{i,k\}} \mathrm{E}(\pi_j)\,\mathrm{E}(\pi_\ell)\,\mathrm{Cov}(Z_\ell, Z_j | Z_i = z_i) \\
0 &< \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i) \\
&= \mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)\,\mathrm{Cov}(Z_j, Z_k | Z_i = z_i) + \sum_{\ell \notin \{i,j\}} \mathrm{E}(\pi_k)\,\mathrm{E}(\pi_\ell)\,\mathrm{Cov}(Z_\ell, Z_k | Z_i = z_i)
\end{aligned}
$$

These inequalities can be reformulated in terms of bounds on the covariance of Z_j and Z_k.[23]

$$
-\sum_{\ell \notin \{i,j\}} \frac{\mathrm{E}(\pi_\ell)}{\mathrm{E}(\pi_j)} \mathrm{Cov}(Z_\ell, Z_k | Z_i = z_i) < \mathrm{Cov}(Z_j, Z_k | Z_i = z_i) < -\sum_{\ell \notin \{i,k\}} \frac{\mathrm{E}(\pi_\ell)}{\mathrm{E}(\pi_k)} \mathrm{Cov}(Z_\ell, Z_j | Z_i = z_i)
$$

Therefore, the restriction that the weights $\{\omega_j\}_{j \neq i}$ are non-negative becomes equivalent to the condition that $\mathrm{Cov}(Z_j, Z_k | Z_i = z_i)$ does not satisfy the inequalities above for any $j, k \neq i$. $\square$

Having proven this claim, our next step is to write down an expression for the IV estimand as a weighted average of individual β_i realizations. We establish the following decomposition:

$$
\begin{aligned}
\beta_i^{\mathrm{IV}}(z_i) = \frac{\sum_{j \neq i} \mathrm{E}(\pi_j) \times \mathrm{Cov}(Y_i, Z_j | Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} &= \frac{\sum_{j \neq i} \mathrm{E}(\pi_j) \times \left(\sum_{\ell \neq i} \mathrm{E}(\beta_i \pi_\ell) \times \mathrm{Cov}(Z_\ell, Z_j | Z_i = z_i) \right)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} \\
&= \frac{\sum_{\ell \neq i} \mathrm{E}(\beta_i \pi_\ell) \times \left(\sum_{j \neq i} \mathrm{E}(\pi_j) \times \mathrm{Cov}(Z_\ell, Z_j | Z_i = z_i) \right)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} \\
&= \frac{\sum_{\ell \neq i} \mathrm{E}(\beta_i \pi_\ell) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_\ell | Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} \\
&= \mathrm{E}\left(\beta_i \times \frac{\sum_{\ell \neq i} \pi_\ell \times \mathrm{Cov}(\bar{Y}_{-i}, Z_\ell | Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} \right)
\end{aligned}
$$

To obtain the second equation above, we switch the summation order in the numerator. The final equation follows from linearity of expectation. Expressed as an integral, $\beta_i^{\mathrm{IV}}(z_i)$ equals:

$$
\begin{aligned}
\beta_i^{\mathrm{IV}}(z_i) &= \int_{\mathrm{supp}(\beta_i)} \beta_i \times \omega(\beta_i | z_i) d\beta_i, \\
\text{where:} \quad \omega(\beta_i | z_i) &= \frac{\sum_{\ell \neq i} \mathrm{E}(\pi_\ell | \beta_i) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_\ell | Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} f_{\beta_i}(\beta_i)
\end{aligned}
$$

[23] To see how, divide both inequalities by $\mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)$, which we assume is positive without loss of generality. If $\mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)$ is negative, then the inequalities flip, and the claim still holds as j and k are chosen arbitrarily.

The final step of our proof is to demonstrate that the weights $\omega(\beta_i|z_i)$ are all non-negative as long as Assumptions PM and NNW are satisfied. We provide a proof of this claim below.

Claim 2. Under Assumptions PM and NNW, $\omega(\beta_i|z_i)$ is non-negative for every value of β_i.

Proof. Using Proposition 1, we can write the coefficient π_j, for any $j \neq i$, as follows:

$$\pi_j = \begin{cases} \gamma_j \times \dfrac{\prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right)}{(|\mathcal{N}|-1) \times \det(I - \mathbf{B})} & \text{if } j \in \mathcal{N} \\ 0 & \text{if } j \notin \mathcal{N} \end{cases}$$

Under Assumption II, $\prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right) > 0$ and $\det(I - \mathbf{B}) > 0$ with probability one. Also, by PM, either $\gamma_j \geq 0$ with probability one or $\gamma_j \leq 0$ with probability one. Without loss of generality, assume that $\gamma_j \geq 0$ with probability one. Then $\mathrm{P}(\pi_j \geq 0) = 1$, so it follows that:

$$\begin{aligned} \mathrm{E}(\pi_j) &= \int_{-\infty}^{\infty} \pi_j f_{\pi_j}(\pi_j) d\pi_j = \int_0^{\infty} \pi_j f_{\pi_j}(\pi_j) d\pi_j \geq 0 \\ \mathrm{E}(\pi_j|\beta_i) f_{\beta_i}(\beta_i) &= \int_{-\infty}^{\infty} \pi_j f_{\pi_j|\beta_i}(\pi_j|\beta_i) f_{\beta_i}(\beta_i) d\pi_j = \int_0^{\infty} \pi_j f_{\pi_j,\beta_i}(\pi_j, \beta_i) d\pi_j \geq 0 \end{aligned}$$

These inequalities imply $\mathrm{E}(\pi_j)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ and $\mathrm{E}(\pi_j|\beta_i)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i) f_{\beta_i}(\beta_i)$ are either both non-negative or both non-positive across all $\beta_i \in \mathrm{supp}(\beta_i)$. Moreover, as the index j was chosen arbitrarily, this relationship applies for any chosen index $j \in \{1, \ldots, N\} \backslash i$.

Assumption NNW ensures that $\mathrm{E}(\pi_j)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ has the same sign across all $j \neq i$. Since these terms also share the same sign as $\mathrm{E}(\pi_j|\beta_i)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i) f_{\beta_i}(\beta_i)$, for all interaction effects $\beta_i \in \mathrm{supp}(\beta_i)$ and $j \neq i$, the weights $\omega(\beta_i|z_i)$ must all be non-negative. $\square$

Proof of Proposition 4

As a first step, we decompose the IV estimand to isolate the average of interaction effects:

$$\begin{aligned} \beta_i^{\mathrm{IV}}(z_i) &= \frac{\sum_{j \neq i} \mathrm{E}(\beta_i \pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} \\ &= \mathrm{E}(\beta_i) + \underbrace{\frac{\sum_{j \neq i} \mathrm{Cov}(\beta_i, \pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)}}_{(*)} \end{aligned}$$

Under Assumption NNW, the product $\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ has the same sign across all $j \neq i$. So, whenever $\mathrm{Cov}(\beta_i, \pi_j)$ has the same sign as $\mathrm{E}(\pi_j)$ for all $j \neq i$, the term $(*)$ will be positive. Alternatively, if $\mathrm{Cov}(\beta_i, \pi_j)$ and $\mathrm{E}(\pi_j)$ have opposite signs for all $j \neq i$, then the term $(*)$ will be negative. This reasoning leads us to the second step of the proof, where we show $\psi_i = \frac{1}{N-1} \sum_{j \neq i} \left[\beta_j \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{N-1}\right)\right]$ governs the sign of $\mathrm{Cov}(\beta_i, \pi_j)$ relative to $\mathrm{E}(\pi_j)$.

Pick any j, where $j \neq i$. By the Assumption PM, either $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$. Without loss of generality, assume $\mathrm{P}(\gamma_j \geq 0) = 1$. Then, as shown in the proof of Proposition

3, the average of π_j must be positive. In addition, the Law of Total Covariance implies that:

$$\text{Cov}(\beta_i, \pi_j) = \text{E}\left(\text{Cov}(\beta_i, \pi_j | \gamma_j, \beta_{-i}, \mathcal{N})\right) + \underbrace{\text{Cov}\left(\text{E}(\beta_i | \gamma_j, \beta_{-i}, \mathcal{N}), \text{E}(\pi_j | \gamma_j, \beta_{-i}, \mathcal{N})\right)}_{=0}$$

Note that the second term on the right-hand-side is zero because $\text{E}(\beta_i | \gamma_j, \beta_{-i}, \mathcal{N}) = \text{E}(\beta_i)$. As implied by Proposition 1, the coefficient π_j can be represented in terms of ψ_i as follows:

$$\begin{aligned}\pi_j &= \mathbb{1}\{j \in \mathcal{N}\} \times \frac{\gamma_j \times \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right)}{(|\mathcal{N}| - 1) \times \det(I - \mathbf{B})} \\ &= \mathbb{1}\{j \in \mathcal{N}\} \times \frac{\gamma_j \times \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right)}{(|\mathcal{N}| - 1) \times \left[A_{ii} - \beta_i \times \psi_i / (|\mathcal{N}| - 1)^2\right]}\end{aligned}$$

where A_{ii} depends only on β_{-i} and $\mathcal{N}$. To simplify notation, define the following parameters:

$$\begin{aligned}\delta_{ij} &= \mathbb{1}\{j \in \mathcal{N}\} \times (|\mathcal{N}| - 1) \times \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}| - 1}\right) \\ \xi_i &= (|\mathcal{N}| - 1)^2 \times A_{ii}\end{aligned}$$

These terms δ_{ij} and ξ_i depend only on β_{-i} and $\mathcal{N}$. Also, δ_{ij} is positive with probability one.

Using this new notation, we can now express covariance between β_i and π_j as follows:

$$\begin{aligned}\text{Cov}(\beta_i, \pi_j) &= \text{E}\left(\text{Cov}\left(\beta_i, \frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} \middle| \gamma_j, \beta_{-i}, \mathcal{N}\right)\right) \\ &= \text{E}\left(\text{E}\left(\left[\beta_i - \text{E}(\beta_i)\right] \times \left[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} - \text{E}\left(\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} \middle| \gamma_j, \beta_{-i}, \mathcal{N}\right)\right] \middle| \gamma_j, \beta_{-i}, \mathcal{N}\right)\right) \\ &= \text{E}\left(\text{E}\left(\left[\beta_i - \text{E}(\beta_i)\right] \times \left[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} - \frac{\gamma_j \times \delta_{ij}}{\xi_i - \text{E}(\beta_i) \times \psi_i}\right] \middle| \gamma_j, \beta_{-i}, \mathcal{N}\right)\right) \\ &\quad + \text{E}\left(\text{E}\left(\left[\beta_i - \text{E}(\beta_i)\right] \times \left[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \text{E}(\beta_i) \times \psi_i} - \text{E}\left(\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} \middle| \gamma_j, \beta_{-i}, \mathcal{N}\right)\right] \middle| \gamma_j, \beta_{-i}, \mathcal{N}\right)\right) \\ &= \text{E}\left(\left[\beta_i - \text{E}(\beta_i)\right] \times \left[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} - \frac{\gamma_j \times \delta_{ij}}{\xi_i - \text{E}(\beta_i) \times \psi_i}\right]\right) \\ &= \text{E}\left(\psi_i \times \underbrace{\frac{\gamma_j \times \delta_{ij} \times \left[\beta_i - \text{E}(\beta_i)\right]^2}{(\xi_i - \text{E}(\beta_i) \times \psi_i)(\xi_i - \beta_i \times \psi_i)}}_{\geq 0 \text{ almost surely and } \neq 0 \text{ with positive probability}}\right)\end{aligned}$$

If $\psi_i > 0$ with probability one, then $\text{Cov}(\beta_i, \pi_j) > 0$. Alternatively, if $\psi_i < 0$ with probability one, then $\text{Cov}(\beta_i, \pi_j) < 0$. Therefore, we conclude that $\text{Cov}(\beta_i, \pi_j)$ has the same (opposite) sign as $\text{E}(\pi_j)$ in cases where ψ_i is positive (negative) with probability one. Because j is chosen arbitrarily, this relationship holds for all $j \neq i$. By the arguments above, this property ensures that the term $(*)$ is positive when $\text{P}(\psi_i > 0) = 1$, while $(*)$ is negative when $\text{P}(\psi_i < 0) = 1$.

$\square$

Proof of Proposition 5

Assumption I implies that the reduced-form is well-defined. Under Assumption II, $\det(I-\mathbf{B})$ and all product terms of the form $\prod_{\ell\in S}\left(1+\frac{\beta_\ell}{N-1}\right)$ are strictly positive. To ease notation, let:

$$\nu_i = \prod_{\ell\neq i}\left(1+\frac{\beta_\ell}{N-1}\right) \quad \text{and} \quad \nu_{ij} = \prod_{\ell\notin\{i,j\}}\left(1+\frac{\beta_\ell}{N-1}\right).$$

We first prove part (iii). By Proposition 1, for any $i \neq j$, the spillover effect $\Delta Y_i/\Delta Z_j$ equals:

$$\frac{\Delta Y_i}{\Delta Z_j} = \gamma_j \times \frac{\beta_i \nu_{ij}}{(N-1)\det(I-\mathbf{B})}.$$

Since $\gamma_j > 0$, $\nu_{ij} > 0$, and $\det(I-\mathbf{B}) > 0$, the spillover effect always has the same sign as β_i.

We next prove part (iv). For any distinct i, j, k, the difference $\Delta Y_j/\Delta Z_i - \Delta Y_k/\Delta Z_i$ is:

$$\begin{aligned}
\frac{\Delta Y_j}{\Delta Z_i} - \frac{\Delta Y_k}{\Delta Z_i} &= \frac{\gamma_i}{(N-1)\det(I-\mathbf{B})}\left[\beta_j \prod_{\ell\notin\{i,j\}}\left(1+\frac{\beta_\ell}{N-1}\right) - \beta_k \prod_{\ell\notin\{i,k\}}\left(1+\frac{\beta_\ell}{N-1}\right)\right] \\
&= \frac{\gamma_i \prod_{\ell\notin\{i,j,k\}}\left(1+\frac{\beta_\ell}{N-1}\right)}{(N-1)\det(I-\mathbf{B})}\left[\beta_j\left(1+\frac{\beta_k}{N-1}\right) - \beta_k\left(1+\frac{\beta_j}{N-1}\right)\right] \\
&= \frac{\gamma_i \prod_{\ell\notin\{i,j,k\}}\left(1+\frac{\beta_\ell}{N-1}\right)}{(N-1)\det(I-\mathbf{B})} \times (\beta_j - \beta_k).
\end{aligned}$$

The term multiplying $\beta_j - \beta_k$ is strictly positive, so this quantity has the same sign as $\beta_j - \beta_k$.

We next prove part *(v)*. The total individual effect and the total effect on the average are:

$$\begin{aligned}
\frac{\Delta Y_i}{\Delta Z_i} &= \frac{\gamma_i}{\det(I-\mathbf{B})}\left[\nu_i - \sum_{j\neq i}\frac{\beta_j}{N-1}\nu_{ij}\right] = \frac{\gamma_i \nu_i}{\det(I-\mathbf{B})}\left[1 - \sum_{j\neq i}\frac{\beta_j}{N-1+\beta_j}\right] \\
\frac{\Delta \bar{Y}}{\Delta Z_i} &= \frac{\gamma_i \nu_i}{N\det(I-\mathbf{B})}.
\end{aligned}$$

For every $j \neq i$, Assumption II implies $\frac{\beta_j}{N-1+\beta_j} < \frac{1}{N}$. Hence, $\sum_{j\neq i}\frac{\beta_j}{N-1+\beta_j} < \frac{N-1}{N} < 1$, which means that the bracketed term must be positive. Because $\gamma_i > 0$, $\nu_i > 0$, and $\det(I-\mathbf{B}) > 0$, it follows that $\Delta Y_i/\Delta Z_i > 0$ and $\Delta \bar{Y}/\Delta Z_i > 0$. Thus, the statements in part *(v)* must hold.

Parts *(i)* and *(ii)* follow from the definitions of the social multipliers in (13) and (14), as well as the positivity of the own effects just established. For the agent-specific multiplier:

$$M_{(i)} - 1 = \frac{\sum_{j=1}^{N}\Delta Y_j/\Delta Z_i}{\Delta Y_i/\Delta Z_i} - 1 = \frac{\sum_{j\neq i}\Delta Y_j/\Delta Z_i}{\Delta Y_i/\Delta Z_i} = \frac{(N-1)\Delta \bar{Y}_{-i}/\Delta Z_i}{\Delta Y_i/\Delta Z_i}.$$

Because $\Delta Y_i/\Delta Z_i > 0$, $M_{(i)} - 1$ has the same sign as $\Delta\bar{Y}_{-i}/\Delta Z_i$. For the aggregate multiplier:

$$
\begin{aligned}
M - 1 &= \frac{\sum_{i=1}^{N} \Delta\bar{Y}/\Delta Z_i}{\frac{1}{N}\sum_{i=1}^{N} \Delta Y_i/\Delta Z_i} - 1 \\
&= \frac{\sum_{i=1}^{N} \Delta\bar{Y}/\Delta Z_i - \frac{1}{N}\sum_{i=1}^{N} \Delta Y_i/\Delta Z_i}{\frac{1}{N}\sum_{i=1}^{N} \Delta Y_i/\Delta Z_i} \\
&= \frac{\frac{N-1}{N}\sum_{i=1}^{N} \Delta\bar{Y}_{-i}/\Delta Z_i}{\frac{1}{N}\sum_{i=1}^{N} \Delta Y_i/\Delta Z_i}.
\end{aligned}
$$

The denominator is positive by part *(v)*, so $M - 1$ has the same sign as $\sum_{i=1}^{N} \Delta\bar{Y}_{-i}/\Delta Z_i$. □

A.3. Additional Proofs and Discussion

Network-Based Instruments

Suppose the researcher observes a vector of covariates $X_g = (X'_{1g}, \ldots, X'_{|\mathcal{N}_g|g})'$ and knows the network links within a group. In this case, a network version of equation (1) can be written as:

$$
Y_{ig} = \alpha_{ig} + \frac{\beta_{ig}}{|\mathcal{N}_g| - 1}\sum_{j\neq i} d_{ijg} Y_{jg} + X'_{ig}\gamma_{iig} + \sum_{j\neq i} d_{ijg} X'_{jg}\gamma_{ijg}, \quad \text{for } i \in \mathcal{N}_g,
$$

where $d_{ijg} \in \{0, 1\}$ indicates whether agent j is linked to agent i in group g.[24] The final two terms in this equation map directly into $Z'_{ig}\gamma_{ig}$ in (1). In particular, one can define the vector $Z_{ig} = (X'_{ig}, \{X'_{jg} : d_{ijg} = 1\}_{j\neq i})'$ with corresponding coefficients $\gamma_{ig} = (\gamma'_{iig}, \{\gamma'_{ijg} : d_{ijg} = 1\}_{j\neq i})'$.

The key exclusion is that an agent i's outcome equation may depend directly on her own covariates and on those of linked agents, but not on the covariates of agents with $d_{ijg} = 0$. Therefore, known network exclusions transform the common observable vector X_g into agent-specific shifters Z_{ig}: two agents in the same group can have different Z_{ig} vectors because the known links select different components of X_g. Formally, for any agent i, if the function $h(X_{kg})$ enters Z_{jg} for some j with $d_{ijg} = 1$, but $h(X_{kg})$ is excluded from Z_{ig} because $d_{ikg} = 0$, then $h(X_{kg})$ is is a valid instrument for endogenous peer outcomes in i's outcome equation. Moreover, if $h(X_{kg})$ it has a nonzero direct effect on Y_{jg}, then it is a relevant instrument. In a case where all agents are linked to all others, the construction yields no excluded variation.

[24] In constant-effects network models, the endogenous peer term is often divided by $\sum_{j\neq i} d_{ijg}$, the number of agents linked to agent i. Since β_{ig} is unrestricted, this alternative normalization can be absorbed into β_{ig}.