The Linear-in-Means Model with Heterogeneous Interactions*

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Abstract

We study peer effects in linear-in-means models with heterogeneous interaction effects. The classical linear-in-means model imposes strict homogeneity on the interaction effects, yielding testable implications that can be readily examined in data. We relax these restrictions to allow for both positive and negative interaction effects that vary within and across groups. This extension makes the linear-in-means model suited to study a wide range of economic behaviors in addition to peer effects, including joint labor supply decisions within households and strategic interactions among firms. We analyze what can and cannot be learned from frequently used OLS and IV estimands for linear-in-means models under heterogeneous interaction effects. While these estimands do not lead to point identification, they can still be used to draw inferences about key economic quantities. We apply these results to two economic applications: classroom peer effects in Kenyan primary schools and strategic pricing decisions among cocoa traders in Sierra Leone. In each application, we reject homogenous interaction effects. Yet, we still draw meaningful inferences about endogenous interactions and social multipliers while allowing for heterogeneous interaction effects.

1 Introduction

Peer effects models are widely used in economics to study how individuals' actions are shaped by those around them, with applications ranging from education and health to labor markets and beyond. The classical linear-in-means model (Manski, 1993) remains the most commonly used framework for empirically analyzing these interactions.[1] This model typically assumes strict homogeneity in the endogenous interaction effects, requiring that all individuals, within

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[1] Applications of the linear-in-means framework include Sacerdote (2001), Guryan et al. (2009), Patacchini & Zenou (2009), Duflo et al. (2011), Dahl et al. (2014), and Casaburi & Reed (2022), among others. Blume et al. (2015) and Boucher et al. (2024) study the microfoundations and economic properties of these models.

and across peer groups, are influenced in exactly the same way by the average outcome of their peers. Identification and estimation is well-studied under this homogeneity assumption, with researchers typically relying on linear OLS and IV estimators to recover economic quantities of interest (Kline & Tamer, 2020). However, the identification arguments behind these estimators do not readily transfer to settings with heterogeneous interaction effects.

The goal of our paper is to study peer effects in linear-in-means models with heterogeneity in endogenous interaction effects. Our main contribution is to analyze what can and cannot be learned from frequently used OLS and IV estimands for linear-in-means models. Although these estimands do not generally lead to point identification under heterogeneous effects, we show that they still offer valuable insights into key economic quantities—even in cases where the available instruments are binary or have limited support. This stands in contrast to the existing work on identification under heterogeneous effects, such as in Masten (2017), which place strong demands on the available instruments and can often be difficult to implement.

We consider a setting with two or more groups, where each group g consists of a set of agents $\mathcal{N}_g$. Each agent i in group g has an outcome Y_{ig}, which is influenced by the outcomes of other agents in the same group. This interdependence is characterized by a linear system:

$$Y_{ig} = \alpha_{ig} + \frac{\beta_{ig}}{|\mathcal{N}_g| - 1} \sum_{j \neq i} Y_{jg} + Z_{ig}' \gamma_{ig}, \quad \text{for } i \in \mathcal{N}_g.$$

In these equations, $\{\alpha_{ig}\}_{i \in \mathcal{N}_g}$, $\{\beta_{ig}\}_{i \in \mathcal{N}_g}$, and $\{\gamma_{ig}\}_{i \in \mathcal{N}_g}$ are all unknown structural parameters. Additionally, $\{Z_{ig}\}_{i \in \mathcal{N}_g}$ is a set of observed variables, which could include individual-level shifters, if $Z_{ig} \neq Z_{jg}$ for $i \neq j$, as well as group-level covariates, if $Z_{ig} = Z_{jg}$ for all $i, j \in \mathcal{N}_g$.

In this model, the parameter β_{ig} represents the individual interaction effect, indicating how each agent i in group g is influenced by the average outcome in the rest of the group. Whereas the classical linear-in-means model maintains that β_{ig} is constant across individuals i and groups g, we allow the interaction effects to differ along both these dimensions. Also, unlike previous work, we do not restrict the sign or magnitude of β_{ig}. Therefore, agents may be positively or negatively affected, however intensely, by their peers. The parameters α_{ig} and γ_{ig} specify how the variables Z_{ig} would determine an agent i's outcome Y_{ig} in absence of spillover effects. We allow these terms to vary freely among agents within and across groups. We also do not restrict the size or composition of each group, as characterized by the set $\mathcal{N}_g$.

In Section II, we begin by reviewing the economic quantities commonly studied in models with constant effects, along with the identification strategies used to recover these quantities. To guide and interpret our results, we draw on three examples: classroom peer effects, household labor supply decisions, and competition among firms in oligopolies. In each example, we show that assuming constant effects imposes strong restrictions on individual preferences or technology, whereas allowing for heterogeneous effects relaxes these restrictions and allows us to study a richer set of economic questions. We also show that the constant effects model yields testable implications in the form of over-identification tests and restrictions on OLS estimands, which can be used to assess whether agents have homogeneous interaction effects.

Motivated by this analysis, we consider, in Section III, the heterogeneous effects model, which allows α_{ig}, β_{ig}, and γ_{ig} to vary freely among agents within and across groups. Under this more general framework, we derive new expressions for the equilibrium outcomes in terms of the individual interaction effects. We use these expressions to characterize equilibrium behavior in the presence of heterogeneity, revealing how different configurations of interaction effects distort group-level outcomes. We find that, with heterogeneous effects, the equilibrium impact of an exogenous shock on group-level outcomes depends on which agents in the group are directly exposed to that shock. These equilibrium effects may also differ across groups.

We then investigate what features of the model are recovered from OLS and IV estimation under heterogeneous effects. We start by analyzing a class of OLS estimands obtained by regressing the outcomes Y on exogenous variables Z (or linear combinations of Z). We show that correctly specified OLS regressions can recover the average equilibrium effects of Z on Y across groups, even when interaction effects are heterogeneous. These regressions also shed light on social multiplier effects, which measure how network spillovers distort the impact of individual-level shocks on group-level outcomes (Glaeser et al., 2003). Under heterogeneous effects, OLS does not lead to point identification of social multipliers. Yet, we show that OLS can still be used to test for positive (or negative) multipliers, allowing us to learn whether spillovers tend to amplify (or suppress) the impacts of targeted policies. Moreover, we show that OLS can be used to test for the presence of positive (or negative) interaction effects.

Next, we analyze what economic quantities are recovered from IV estimation. We study a large class of IV estimands that use exclusion restrictions to recover the interaction effects β_{ig}. We show that, with heterogeneous effects, the IV estimand represents a particular weighted average of interaction effects, which places higher weight on groups where aggregate outcomes are more responsive to the instruments. We then derive necessary and sufficient conditions for these weights to be non-negative, which we view as a minimal requirement for the IV estimand to be informative about interaction effects. We also show how the IV estimand compares to an unweighted average of interaction effects. In general, this relationship depends on the signs of the interactions, whether they are positive or negative. We prove that in many common network settings, such as classical peer effects, oligopoly models, and public goods games, the IV estimand will necessarily overstate the average interaction effect.

In Section IV, we apply our analysis to data for two economic applications that employ the linear-in-means model with constant effects: peer effects in Kenyan grade schools (Duflo et al., 2011) and competition between cocoa traders in Sierra Leone (Casaburi & Reed, 2022). In both instances, we find strong evidence to reject homogeneous interaction effects. In the first application, we find that peer effects differ across classrooms. In the second application, we conclude that traders respond strategically in different ways to their competitors' actions.

Given these findings, we then re-analyze our two empirical applications under the linear-in-means model with heterogeneous interaction effects. In the Kenyan primary school setting, we find that peer effects are positive for a large share of students. Our estimate of the upper bound on the average peer effect implies that a 1 point increase in peers' average test scores raises a student's own test score by no more than 0.45 points, on average. We also find strong evidence of positive social multiplier effects, indicating that in some classrooms, the impact of policies targeting individual achievement is amplified through peer interactions. In the analysis of competition between cocoa traders in Sierra Leone, we find evidence of strategic interactions and imperfect competition in price setting. Our estimated upper bound on the average conduct parameter implies that increasing competitors' cocoa purchases by 1 pound reduces a trader's own purchases by no more than 0.02 pounds, on average. We find no evidence of social multiplier effects in this setting, suggesting that strategic interactions do not substantially alter how trader-specific changes in demand or costs affect total market output.

Our paper contributes to two literatures. First, we contribute to a literature on the empirical analysis of social interactions; see Paula (2017) and Kline & Tamer (2020) for recent surveys.[2] Within this literature, there is increasing recognition of the importance of accounting for individual heterogeneity in endogenous interaction effects.[3] While economic theory is well-studied in these cases (Jackson & Zenou, 2015), there is less work addressing the identification of models with heterogeneous interaction effects. One key exception is Masten (2017), who studies identification for a linear peer effects model with random coefficients.[4] He proves that the marginal distributions of the coefficients are point identified if there is an instrument with continuous variation over a large support. However, he also shows that instruments are insufficient for recovering the full joint distribution of random coefficients. These results raise questions about what can be learned about other economic quantities, such as equilibrium effects and social multipliers, in the presence of heterogeneity. Our paper addresses this question by analyzing how to interpret and learn from OLS and IV estimation in contexts with heterogeneous interaction effects. We view our results as constructive. While point identification might not be achievable, we find that meaningful inferences can still be made from frequently used OLS and IV estimators. Our approach is broadly applicable for a variety of settings where access to a continuous instrument with large support is not feasible.

The second literature to which we contribute is concerned with the interpretation of linear OLS and IV estimands in settings with unobserved heterogeneity in treatment effects. Mogstad & Torgovitsky (2024) give a recent survey of this work. In a seminal paper, Imbens & Angrist (1994) pioneer a framework for interpreting linear IV estimands as weighted aver-

[2]See Blume et al. (2011) for more discussion. Also, Sacerdote (2011) surveys the literature on peer effects in education, and Browning et al. (2014) discusses the use of social interaction models for household behavior.

[3]Sacerdote (2011) highlights the importance of allowing for heterogeneity in interaction effects. Using a discrete choice model, Volpe (2025) finds robust evidence that these effects differ across demographic groups.

[4]Hurwicz (1950), Kelejian (1974), and Hahn (2001) also examine simultaneous equations with random coefficients. Hurwicz (1950) does not give explicit identification results. In addition, as Masten (2017) points out, Kelejian (1974) and Hahn (2001) conduct analyses that are based on self-contradictory assumptions.

ages of local average treatment effects, and Angrist et al. (2000) extend these interpretations to supply and demand models consisting of two simultaneous equations. The system of linear simultaneous equations for peer effects differs in two important ways from the linear supply and demand system studied by Angrist et al. (2000). First, the supply and demand system is restricted to a network of two agents: a representative firm and a representative consumer in each market. Second, the supply and demand system focuses on specific interaction effects where the sign is known, i.e., upward-sloping supply and downward-sloping demand. In contrast, the system of linear simultaneous equations we consider does not place restrictions on the signs of the interaction effects, which means that agents' outcomes could be strategic substitutes and/or complements. Therefore, we can apply our model to a wide range of settings that involve substitutabilities and/or complementarities in decision-making, including peer effects, household labor supply decisions, and competition among firms in an oligopoly.

Our paper contributes to this literature by demonstrating how to interpret linear OLS and IV estimands for linear peer effects models with heterogeneous interaction effects. Our analysis finds that many of the existing tools for interpreting these estimands do not easily transfer to peer effects models. For example, with peer groups larger than two, the standard monotonicity conditions for IV to have a causal interpretation (Imbens & Angrist, 1994) place strong restrictions on the peer effects, which are unlikely to apply in many practical settings. We propose alternative, weaker conditions under which IV retains a causal interpretation. We then demonstrate how this causal parameter allows us to learn about economic quantities of interest. Overall, our analysis gives an accessible framework for learning about heterogeneous interaction effects and social multipliers from frequently used linear OLS and IV estimands.

2 The Linear-in-Means Model

In this section, we present the linear-in-means model, provide economic interpretations, and define a set of target parameters. We then explain how each of these parameters is recovered from the data under the assumption that the endogenous interaction effects are homogeneous.

2.1 Econometric Model

In its general form, the linear-in-means model with heterogeneous interactions is given by:

$$Y_{ig} = \alpha_{ig} + \frac{\beta_{ig}}{|\mathcal{N}_g| - 1} \sum_{j \neq i} Y_{jg} + Z_{ig}' \gamma_{ig}, \quad \text{for } i \in \mathcal{N}_g. \tag{1}$$

To interpret the flexibility of this model, it is useful to contrast it with the classical linear-in-means model, which assumes that all the interaction effects are homogeneous. Specifically, the classical model imposes that $\beta_{ig} = \beta_{jg}$ for any two agents i and j, which means that all agents in a group are affected in the exact same way by their peers. Additionally, it requires that $\beta_{ig} = \beta_i$ for all i and g, implying that every group exhibits identical interaction effects.

One notable implication of these homogeneity restrictions is that all interaction effects must have the same sign. For instance, it is generally assumed that $\beta_{ig} \geq 0$ for every agent i and group g. This assumption imposes uniform strategic complementarities, where everyone in the population seeks to conform to the mean outcome of their peers. Situations where some agents prefer to conform while others prefer to deviate are hence ruled out by construction.

In addition to these homogeneity restrictions, the classical linear-in-means model generally assumes that $|\beta_{ig}| < 1$ for all i and g, which ensures that the interaction effects are small in magnitude. Also, while there are many variants of this model, many papers maintain that the coefficient γ_{ig} is homogeneous, which means that the direct effect of Z_{ig} on Y_{ig} in absence of spillovers is fixed in the population. We summarize these restrictions below for reference.

Classical Linear-in-Means Assumptions

C.1 *(Homogeneous Interactions within Groups).* $\beta_{ig} = \beta_{jg}$ for any two agents $i, j \in \mathcal{N}_g$.
C.2 *(Homogeneous Interactions across Groups).* $\beta_{ig} = \beta_i$ for all agents i and groups g.
C.3 *(Bounded Interaction Effects).* $|\beta_{ig}| < 1$ for all agents i and groups g.
C.4 *(Homogeneous Incidence of Z).* $\gamma_{ig} = \gamma$ for all agents i and groups g.

2.2 Economic Interpretations of the Model

We now illustrate how the linear-in-means model can be derived as the estimating equation for three economic decision problems: peer effects in schools, joint labor supply decisions in households, and strategic interactions among firms in oligopolistic markets. In each example, strong restrictions on the preferences or technology are needed in order to justify the classical linear-in-means assumptions. Relaxing these assumptions therefore makes the model better suited for studying economic behavior in these different settings. Throughout the remainder of the paper, we will continue to draw on these examples to guide and interpret our analysis.

2.2.1 Peer Effects

Consider a peer group g, where each individual i makes a choice Y_{ig} from an action space $\mathbb{R}$. When making their choices, individuals either seek to conform to or deviate from the average behavior of their peers. These social pressures directly enter into each agent's utility function.

$$U_{ig}(Y_{ig}|Z_{ig}, \bar{Y}_{-ig}) = \big(\alpha_{ig} + Z_{ig}'\gamma_{ig}\big)Y_{ig} - \frac{\beta_{ig}}{2}(Y_{ig} - \bar{Y}_{-ig})^2 - \frac{1-\beta_{ig}}{2}Y_{ig}^2.$$

This utility specification is commonly used in the education literature to study peer effects; see Blume et al. (2015) and Kline & Tamer (2020). The first component of utility captures the non-social determinants of an agent's choice. The second term represents social pressure, penalizing the squared deviations between an agent's own choice and the average choice of her peers. The third term is a convex cost of action. In this framework, the social interaction effect β_{ig} determines the extent to which the agent seeks to conform to or diverge from peer behavior. In equilibrium, agents' optimal decisions $\{Y_{ig}\}_{i\in\mathcal{N}_g}$ will satisfy the equations (1).[5]

The classical linear-in-means assumptions would imply that all individuals experience the same amount of social pressure, leading to identical marginal rates of substitution between private and social utility. By relaxing these assumptions, we allow individuals to face different types of social pressure. For example, it could be that certain agents seek to deviate from, rather than conform to, their peers; or, it could be that all agents wish to conform but some do so more than others. Our extension allows for such nuances in the study of peer effects.

2.2.2 Household Labor Supply

Consider a non-unitary model of household labor supply, as discussed in the survey by Donni & Chiappori (2011). Each member of a household allocates a fixed time endowment T between labor and leisure. Let h_{ig} be the number of hours that member i of household g chooses to work, and let W_{ig} be the wage. The resulting labor income for individual i is $Y_{ig} = W_{ig}h_{ig}$.

Members of each household pool their incomes. These incomes are then redistributed so that each member i receives a fraction $\kappa_{ig} \in [0, 1]$ to spend on personal consumption. The total value of household consumption, denoted by C_g, cannot exceed total household income. In addition to consuming $\kappa_{ig}C_g$, each individual i can also consume non-transferable goods. These goods may come in the form of workplace amenities or social assistance benefits, such as healthcare services that only i can access. The value of these goods to individual i is a_{ig}.

Each individual maximizes welfare through leisure and consumption. The return on each input is marginally decreasing, as represented by the following log-additive utility function.

$$\max_{h_{ig}}\ U_{ig}(h_{ig}|W_{ig}, C_g) = \mu_{ig}\log(T - h_{ig}) + (1-\mu_{ig})\log(a_{ig} + \kappa_{ig}C_g), \quad \text{s.t.} \quad C_g = \sum_{j\in\mathcal{N}_g} W_{jg}h_{jg}.$$

The parameter $\mu_{ig} \in [0, 1]$ reflects individual i's relative preference for leisure over consumption. As long as everyone spends some time working, $h_{ig} \in (0, T)$, an interior solution exists.

$$\begin{aligned} Y_{ig} &= -\frac{\mu_{ig}a_{ig}}{\kappa_{ig}} - \mu_{ig}\sum_{j\neq i} Y_{jg} + (1-\mu_{ig})TW_{ig} \\ &= \underbrace{\alpha_{ig}}_{-\frac{\mu_{ig}a_{ig}}{\kappa_{ig}}} + \underbrace{\frac{\beta_{ig}}{|\mathcal{N}_g| - 1}}_{-\mu_{ig}} \sum_{j\neq i} Y_{jg} + \underbrace{\gamma_{ig}}_{(1-\mu_{ig})T} W_{ig}, \quad \text{for } i \in \mathcal{N}_g. \end{aligned}$$

These equilibrium equations satisfy the linear-in-means model (1) where the interaction effect β_{ig} equals $-\mu_{ig}(|\mathcal{N}_g| - 1)$. This interaction effect governs how much an individual i's income falls when the other household members more. It also determines the elasticity of i's earnings with respect to the wage rate W_{ig}. The variable Z_{ig} can be anything that influences i's wage.

[5] An alternative utility specification, used by Epple & Romano (1998) and Calvó-Armengol et al. (2009), is $U_{ig}(Y_{ig}|Z_{ig}, \bar{Y}_{-ig}) = \big(\alpha_{ig} + Z_{ig}'\gamma_{ig}\big)Y_{ig} + \beta_{ig}\bar{Y}_{-ig}Y_i - \frac{1}{2}Y_{ig}^2$, which also rationalizes the linear-in-means model.

The classical linear-in-means assumptions require that the marginal rate of substitution between consumption and leisure is the same across all individuals, both within and between households. By allowing for heterogeneous interaction effects, we permit individuals to make different consumption-leisure trade-offs. For example, we allow labor supply responses to differ between the primary and secondary earners in a household. These responses might also vary across households due to contextual factors such as the number of children in the home.

2.2.3 Firm Oligopoly

Finally, consider a model of oligopolistic competition where firms face heterogeneous, convex cost curves. Following Bresnahan (1981) and Perry (1982), we examine a framework that nests both Bertrand and Cournot competition. This framework assumes that firms form conjectures about their competitors' actions, which are consistent with equilibrium outcomes.

Each market g contains multiple firms i, each producing output Y_{ig}. The price that clears the market is given by an inverse demand: $P_g = a_g - b_g \sum_{i \in \mathcal{N}_g} Y_{ig}$, where a_g and b_g can vary across markets g. A firm's production costs are given by $c_{ig}(Y_{ig}) = (\lambda_{ig0} + Z_{ig}' \lambda_{ig1}) Y_{ig} + \frac{1}{2} \delta_{ig} Y_{ig}^2$, where λ_{ig0}, λ_{ig1}, and δ_{ig} can vary both across firms i and across markets g. Assume that the vector Z_{ig} contains observable cost-shifters, which directly influence the firm's productivity.

We suppose that every firm i has some reference output Y_{ig}^0, which is common knowledge in the market. The firm conjectures that increasing its own output Y_{ig} relative to Y_{ig}^0 causes the other firms to adjust their total output by θ_{ig}, believing that $\sum_{j \neq i} Y_{jg}$ equals $\sum_{j \neq i} Y_{jg}^0 + \theta_{ig}(Y_{ig} - Y_{ig}^0)$. Given these conjectures, each firm i in market g maximizes its profit by solving:

$$\max_{Y_{ig}} \ \Pi_{ig}(Y_{ig} | Z_{ig}, \{Y_{jg}^0\}_{j \neq i}) = P_g Y_{ig} - c_{ig}(Y_{ig}), \quad \text{s.t.} \quad \begin{aligned} P_g &= a_g - b_g \sum_{i \in \mathcal{N}_g} Y_{ig} \\ \sum_{j \neq i} Y_{jg} &= \sum_{j \neq i} Y_{jg}^0 + \theta_{ig}(Y_{ig} - Y_{ig}^0) \\ c_{ig}(Y_{ig}) &= (\lambda_{ig0} + Z_{ig}' \lambda_{ig1}) Y_{ig} + \frac{1}{2} \delta_{ig} Y_{ig}^2. \end{aligned}$$

In this model, θ_{ig} represents the conjectural variation, measuring a firm i's perceived influence in market g. Three special cases are particularly notable. First, if $\theta_{ig} = 0$ for all i, then the model corresponds to Cournot oligopoly. In this case, firms do not internalize the effect of their own output decisions on the behavior of other firms. Second, if $\theta_{ig} = -1$ for all i, then the model is one of Bertrand competition. Here, firms expect that their actions have no effect on total market output. Third, if $\theta_{ig} = |\mathcal{N}_g| - 1$ for all i, then the market is monopolistic. In this setting, each firm acts as if it fully controls the market, which leads to perfect collusion. Given this range of possibilities, it seems natural to permit θ_{ig} to be between -1 and $|\mathcal{N}_g| - 1$.

In equilibrium, each firm's output Y_{ig} must equal its reference output Y_{ig}^0. The resulting equilibrium condition will generate the linear-in-means model (1) as an estimating equation.

$$
\begin{aligned}
Y_{ig} &= \frac{1}{\delta_{ig} + b_g(2+\theta_{ig})} \left[a_g - \lambda_{ig0} - b_g \sum_{j \neq i} Y_{jg} - Z'_{ig}\lambda_{ig1} \right] \\
&= \underbrace{\alpha_{ig}}_{\frac{a_g - \lambda_{ig0}}{\delta_{ig}+b_g(2+\theta_{ig})}} + \underbrace{\frac{\beta_{ig}}{|\mathcal{N}_g| - 1}}_{-\frac{b_g}{\delta_{ig}+b_g(2+\theta_{ig})}} \sum_{j \neq i} Y_{jg} + Z'_{ig} \underbrace{\gamma_{ig}}_{-\frac{\lambda_{ig1}}{\delta_{ig}+b_g(2+\theta_{ig})}}, \quad \text{for } i \in \mathcal{N}_g.
\end{aligned}
$$

To understand how the classical linear-in-means assumptions restrict firm behavior, consider the interaction effect $\beta_{ig} = -\frac{b_g(|\mathcal{N}_g|-1)}{\delta_{ig}+b_g(2+\theta_{ig})}$, which represents a firm-specific conduct parameter, as defined by Weyl & Fabinger (2013). This quantity measures how a firm's output responds to the output of its competitors, and it depends on three factors: the elasticity of consumer demand b_g, the slope δ_{ig} of the marginal cost curve, and the conjectural variation θ_{ig}. By assuming constant interaction effects, the classical linear-in-means model implicitly requires that: (1) consumer demand is equally elastic in every market, (2) firms' marginal costs have the same curvature, and (3) all firms share the same beliefs about competition. By extending the model to allow for heterogeneous interaction effects, we relax each of these restrictions.

2.3 Economic Quantities of Interest

Depending on the empirical context, researchers may be interested in learning about a range of reduced form and structural parameters in the model. In Table 1, we list several economic quantities that are commonly studied in the classical linear-in-means model. For each one, we give a definition and derive its expression in terms of the model's structural parameters. To ease notation, we suppress group subscripts and set $\mathcal{N} = \{1, \ldots, N\}$, while noting that the group size can freely vary. Also, for expositional purposes, we assume Z_{ig} is one-dimensional, although including a vector of shifters/covariates does not meaningfully impact our analysis.

For now, we analyze the economic quantities under the classical linear-in-means model, as presented in Column 2 of Table 1, and defer the more general analysis with heterogeneous effects (Column 3 of Table 1) to Section III. Under Assumption C.3, the system of equations (1) exhibits a unique solution, which allows us to derive the following reduced form equations:

$$
Y_i = \left(1 + \beta_i \frac{1}{N-1} \sum_{j \neq i} \psi_{ji} \right) (\alpha_i + \gamma_i Z_i) + \sum_{j \neq i} \psi_{ij} (\alpha_j + \gamma_j Z_j), \quad \text{for } i \in \{1, \ldots, N\}, \tag{2}
$$

These equations characterize how Z affects Y in equilibrium, after accounting for spillovers. The term ψ_{ij} is a structural parameter representing the equilibrium effect of a unit increase in agent j's outcome Y_j on agent i's outcome Y_i. In Section III, we derive a general expression for ψ_{ij} in terms of the interaction effects $\{\beta_i\}_{i=1}^N$. However, under Assumptions C.1 and C.2, the interaction effects are homogeneous, in which case ψ_{ij} reduces to a constant ψ, given by:

$$\psi = \frac{\beta}{(1-\beta)(N-1+\beta)}.$$

Note that ψ has the same sign as β, and it tends to zero as the group size N tends to infinity. Together with Assumption C.4, the reduced form equations (2) simplify in the following way:

$$Y_i = (1+\beta\psi)(\alpha_i + \gamma Z_i) + \psi \sum_{j\neq i}(\alpha_j + \gamma Z_j), \quad \text{for } i \in \{1,\ldots,N\}. \tag{3}$$

Spillover Effect

The first term in Table 1 is the individual spillover effect of Z_j on Y_i. In a peer effects model, this quantity measures how a student i is indirectly influenced by factors that alter another student j's achievement. In a model of household labor supply, it measures how a person i's income is affected by the wage earned by another family member j. In a model of oligopoly, it measures how the output of a firm i reacts to a productivity shock within another firm j.

The spillover effect may be decomposed as the product of two terms, γ_j and ψ_{ij}, where γ_j denotes the direct effect of Z_j on Y_j, and ψ_{ij} represents the effect of Y_j on Y_i in equilibrium.

$$\frac{\Delta Y_i}{\Delta Z_j} = \gamma_j \times \psi_{ij}. \tag{4}$$

Under Assumptions C.1-C.4, both γ_j and ψ_{ij} are constant across all agent pairs (i,j). Thus, the classical linear-in-means model assumes that the spillover effect of Z_j on Y_i is homogeneous: it does not depend on who receives the direct shock or who is indirectly affected by it.

Total Individual Effect

The second term in the table is the total individual effect of Z_i on Y_i, after accounting for spillovers. We decompose this term to distinguish between direct and indirect effects of Z_i.

$$\frac{\Delta Y_i}{\Delta Z_i} = \gamma_i + \underbrace{\beta_i \frac{\Delta \bar{Y}_{-i}}{\Delta Z_i}}_{\text{Indirect Effect}}, \quad \text{where} \quad \frac{\Delta \bar{Y}_{-i}}{\Delta Z_i} = \gamma_i \times \frac{1}{N-1}\sum_{j\neq i}\psi_{ji}. \tag{5}$$

The indirect effect accounts for network distortions. It depends on the cycles in the network, which specify how an agent's behavior is reflected back onto itself via interactions with others. This feedback loop may either reinforce or undermine the direct effect of the variable Z_i.

To understand when the interaction effects will amplify or suppress the impact Z_i on Y_i, we must examine the product of β_i and $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$. This product represents an indirect interaction effect that agent i has with herself, as measured by evaluating all the cycles in the network that start and end with agent i. If $\beta_i \times \frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$ is positive, then agent

i's behavior is self-reinforcing. In this case, the interaction effects magnify the impact of an exogenous shock: $|\Delta Y_i/\Delta Z_i| > |\gamma_i|$. Conversely, if $\beta_i \times \frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$ is negative, then agent i's actions are self-undermining, thereby suppressing the impact of a shock: $|\Delta Y_i/\Delta Z_i| < |\gamma_i|$.

With Assumptions C.1-C.4, the total individual effect is constant: $\Delta Y_i/\Delta Z_i = \gamma(1+\beta\psi)$. In other words, all agents are affected uniformly by a shock to their own outcome. Moreover, as the product $\beta_i \times \frac{1}{N-1}\sum_{j\neq i}\psi_{ji} = \beta\times\psi$ is non-negative, the classical linear-in-means model assumes that the interaction effects always amplify the impact of a shock: $|\Delta Y_i/\Delta Z_i| \geq |\gamma_i|$. For example, in a peer effects model, social pressure must amplify the impact of additional effort on a student's own academic performance. In a household labor supply model, second earner effects must amplify the impact of receiving a raise on an individual's own income. In a model of firm oligopoly, strategic interactions must amplify the impact of a productivity shock on a firm's own output. As we discuss in Section III, this amplification pattern need not hold under heterogeneous interaction effects. In particular, we show that heterogeneous interactions can, in certain cases, suppress the impact of a shock on an agent's own outcome.

Total Effect on the Average

Third, we define the total equilibrium effect of Z_i on the average outcome $\bar{Y}$ in the group.

$$\frac{\Delta \bar{Y}}{\Delta Z_i} = \frac{1}{N}\left[\frac{\Delta Y_i}{\Delta Z_i} + \sum_{j\neq i}\frac{\Delta Y_j}{\Delta Z_i}\right] = \frac{1}{N}\left[1 + \left(1 + \frac{\beta_i}{N-1}\right)\sum_{j\neq i}\psi_{ji}\right]\times\gamma_i. \tag{6}$$

With Assumptions C.1-C.4, this effect reduces to a constant: $\Delta\bar{Y}/\Delta Z_i = \frac{1}{N}\left(\frac{\gamma}{1-\beta}\right)$. So, in the classical linear-in-means model, the total effect of a shock to agent i on the average outcome $\bar{Y}$ is the same regardless of which agent i is considered. Moreover, because $\Delta\bar{Y}/\Delta Z_i$ has the same sign as γ, any increase in one agent's outcome Y_i must also raise the average outcome $\bar{Y}$.

Social Multiplier Effect

The fourth parameter we define is the social multiplier effect. We use this quantity to measure how network externalities distort the effect of exogenous shocks on aggregate outcomes in a group. Much of the literature on social multipliers (e.g., Goldin & Katz, 2002; Glaeser et al., 2003; Becker & Murphy, 2003) assumes that the interaction effects are positive: $\beta_i \geq 0$ for all i. Under this assumption, network spillovers always amplify the impact of a policy shock on group outcomes. However, this pattern need not hold in settings with negative interaction effects. In such cases, network spillovers have a potential to suppress the impact of a policy.

For the classical linear-in-means model, Glaeser et al. (2003) define the social multiplier to be the ratio of aggregate coefficients to individual coefficients in the reduced form, given by:

$$M^{\text{constant}} = \frac{\Delta\bar{Y}/\Delta\bar{Z}}{\Delta Y_i/\Delta Z_i} = \frac{\beta + N - 1}{\beta + (1-\beta)(N-1)}. \tag{7}$$

This parameter measures how the equilibrium impact of Z on Y changes at different levels of aggregation. As the size of the group N grows large, the multiplier effect tends to $(1-\beta)^{-1}$.

Table 1: Economic Quantities of Interest

Economic Quantity	**Structural Interpretation**	
	Constant Effects	Heterogeneous Effects
Reduced Form Quantities		
Spillover Effect $(\Delta Y_i/\Delta Z_j)$	$\frac{\beta\gamma}{(1-\beta)(N-1+\beta)}$	$\frac{\beta_i\gamma_j\nu_{ij}}{(N-1)\det(I-\mathbf{B})}$
Total Individual Effect $(\Delta Y_i/\Delta Z_i)$	$\gamma+\frac{\beta^2\gamma}{(1-\beta)(N-1+\beta)}$	$\gamma_i+\frac{\beta_i\gamma_i\left(\frac{1}{N-1}\sum_{j\neq i}\beta_j\nu_{ij}\right)}{(N-1)\det(I-\mathbf{B})}$
Total Effect on the Average $(\Delta\bar{Y}/\Delta Z_i)$	$\frac{1}{N}\times\frac{\gamma}{(1-\beta)}$	$\frac{1}{N}\times\frac{\gamma_i\nu_i}{\det(I-\mathbf{B})}$
Individual Social Multiplier $\left(\frac{\sum_{j=1}^N\Delta Y_j/\Delta Z_i}{\Delta Y_i/\Delta Z_i}\right)$	$\frac{\beta+N-1}{\beta+(1-\beta)(N-1)}$	$\frac{\nu_i}{\nu_i-\frac{1}{N-1}\sum_{j\neq i}\beta_j\nu_{ij}}$
Aggregate Social Multiplier $\left(\frac{\sum_{i=1}^N\Delta\bar{Y}/\Delta Z_i}{\frac{1}{N}\sum_{j=1}^N\Delta Y_j/\Delta Z_j}\right)$	$\frac{\beta+N-1}{\beta+(1-\beta)(N-1)}$	$\frac{\frac{1}{N}\sum_{i=1}^N\nu_i\gamma_i}{\frac{1}{N}\sum_{j=1}^N\left(\nu_j-\frac{1}{N-1}\sum_{k\neq j}\beta_k\nu_{jk}\right)\gamma_j}$
Structural Quantities		
No Interference Outcome $(Y_i\|(\bar{Y}_{-i},Z_i)=\mathbf{0})$	α_i	α_i
No Interference Effect $(\Delta Y_i/\Delta Z_i\|\bar{Y}_{-i})$	γ	γ_i
Interaction Effect $(\Delta Y_i/\Delta\bar{Y}_{-i})$	β	β_i
Interaction Effect Correlation $\left(\operatorname{corr}\left(\frac{\Delta Y_i}{\Delta\bar{Y}_{-i}},\frac{\Delta Y_j}{\Delta\bar{Y}_{-j}}\right)\right)$	0	$\operatorname{corr}(\beta_i,\beta_j)$

Notes. The reduced form effects $\Delta Y_i/\Delta Z_j$, $\Delta Y_i/\Delta Z_i$, and $\Delta\bar{Y}/\Delta Z_i$ are defined by holding $\{Z_j\}_{j\neq i}$ fixed. To ease notation in the last column, we let $\nu_i=\prod_{\ell\neq i}\left(1+\frac{\beta_\ell}{N-1}\right)$ and $\nu_{ij}=\prod_{\ell\notin\{i,j\}}\left(1+\frac{\beta_\ell}{N-1}\right)$ for any i and j.

Structural Coefficients and Higher Moments

The next three parameters are the structural coefficients α_i, β_i, and γ_i. Among these terms, the interaction effect β_i is often the primary target parameter (Sacerdote, 2011). In a peer effects model, it measures how much social pressure an individual experiences. In a model of household behavior, it specifies how a person's income depends on the earnings of others. In an oligopoly model, it measures the degree of strategic interaction between firms. The parameters α_i and γ_i also have an economic interpretation, as they indicate how the variable Z_i would impact an agent's outcome Y_i in absence of network interference. In certain settings, it might be important to distinguish between the direct treatment responses and the indirect effects of treatments that arise through social interactions; see Manski (1993) for discussion.

Lastly, we may be interested in the correlation structure of interaction effects for agents in a network—specifically, parameters like $\operatorname{corr}(\beta_i,\beta_j)$ for different individuals i and j. These parameters can offer insights into the formation of network ties. For example, if one individual feels strong pressure to conform to a group, is it likely that her peers would feel similarly?

Do members of the same family share similar preferences over leisure and consumption? Do firms in the same market exhibit similar beliefs about competition? Such questions are unanswerable in the classical linear-in-means model, because the interaction effects are assumed to be homogeneous across all agents. In contrast, the heterogeneous effects framework is well suited for studying these types of correlations and their implications for economic behavior.

2.4 Standard Constant Effects Estimands

We now review the OLS and IV estimands that are typically used to recover reduced form and structural parameters in the classical linear-in-means model. In Table 2, we define these estimands and summarize the economic content that each one delivers, both under the general model with heterogeneous interaction effects and in the homogeneous effects special case.

Table 2: Economic Inferences from OLS and IV Estimands

Economic Quantity	$\beta^{\text{OLS}}(Y_i)$	$\beta^{\text{OLS}}(\bar{Y})$	$\beta^{\text{OLS}}(\bar{Y}_{-i})$	$\beta_i^{\text{OLS}}(\bar{Y})/\beta_i^{\text{OLS}}(Y_i)$	β_i^{IV}
Spillover Effect					
Constant Effects	Ind. Effect	—	Ind. Effect	—	—
Heterogeneous Effects	Avg. Effect	—	—	—	—
Total Individual Effect					
Constant Effects	Ind. Effect	—	—	—	—
Heterogeneous Effects	Avg. Effect	—	—	—	—
Total Effect on the Average					
Constant Effects	—	Ind. Effect	—	—	—
Heterogeneous Effects	—	Avg. Effect	—	—	—
Social Multiplier					
Constant Effects	Test: ≤ 1	—	Test: ≤ 1	Ind. Effect	—
Heterogeneous Effects	—	—	Test: ≤ 1 for all g	—	—
Interaction Effect					
Constant Effects	Test: ≤ 0	—	—	—	Ind. Effect
Heterogeneous Effects	Test: ≤ 0 for all g	—	—	—	Weighted Avg.

Notes. In this table, we define a class of OLS estimands $\beta^{\text{OLS}}(x) = \text{E}(\tilde{Z}\tilde{Z}')^{-1}\,\text{E}(\tilde{Z}x)$, where $\tilde{Z} = (1, Z')'$, and IV estimands $\beta_i^{\text{IV}} = \frac{\text{Cov}(Y_i, \mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})|Z_i=z_i)}{\text{Cov}(\bar{Y}_{-i}, \mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})|Z_i=z_i)}$, where $\tilde{Z}_{-i} = (1, g(Z_{-i}))$ for some monotone function g defined on $\text{supp}(Z_{-i})$. In Column 2, we maintain Assumptions C.1-C.4. In column 3, we maintain Assumptions I-III.

For now, we analyze these estimands under the classical linear-in-means model, imposing Assumptions C.1–C.4, and we postpone the analysis under heterogeneous effects to Section 3. Following the literature, we assume that the vector α is mean independent of the observables Z and group composition $\mathcal{N}$, so that $\mathbb{E}(\alpha|Z, \mathcal{N}) = \mathbb{E}(\alpha)$. We also assume that $\{Z_i\}_{i\in\mathcal{N}}$ are individual-level shifters, and we require that these shifters are not perfectly collinear, which implies that $\text{E}(ZZ')$ is nonsingular. Finally, we assume that $\gamma \neq 0$, so each Z_i has a nonzero effect on agents' outcomes. Under these conditions, each Z_i is a valid instrument in our setup.

Since this model involves simultaneity, instruments play a central role in identification. Specifically, they provide exclusion restrictions, which are factors that directly affect a subset of the agents in a group, while leaving others unaffected. Examples of exclusions are policy variables that shift an agent's marginal cost of action. Alternatively, an exclusion could be a restriction on the interactions in the network, whereby some agents do not directly influence certain members of their group. In the Appendix, we show how to extend the linear-in-means model to reformulate these restrictions as instruments. In doing so, our analysis speaks to a wide range of identification strategies that use exclusions to recover structural parameters.[6]

Frequently-Used OLS Estimands

We begin by analyzing OLS estimands obtained by projecting outcomes Y on individual-level shifters Z. Specifically, we consider three different estimands, $\beta^{\mathrm{OLS}}(Y_i) = \mathbb{E}(\tilde{Z}\tilde{Z}')^{-1}\mathbb{E}(\tilde{Z}Y_i)$, $\beta^{\mathrm{OLS}}(\bar{Y}) = \mathbb{E}(\tilde{Z}\tilde{Z}')^{-1}\mathbb{E}(\tilde{Z}\bar{Y})$, and $\beta^{\mathrm{OLS}}(\bar{Y}_{-i}) = \mathbb{E}(\tilde{Z}\tilde{Z}')^{-1}\mathbb{E}(\tilde{Z}\bar{Y}_{-i})$, which correspond to linear regressions of Y_i, $\bar{Y}$, and $\bar{Y}_{-i}$, respectively, on the vector $\tilde{Z} = (1, Z')'$. Under Assumptions C.1-C.4, these estimands recover the individual spillover effect $\Delta Y_i/\Delta Z_j$, the total individual effect $\Delta Y_i/\Delta Z_i$, and the total effect on the average $\Delta\bar{Y}/\Delta Z_i$. Moreover, the social multiplier effect is identified from a ratio of OLS coefficients, given by $M^{\mathrm{constant}} = \beta_i^{\mathrm{OLS}}(\bar{Y})/\beta_i^{\mathrm{OLS}}(Y_i)$.

As shown in Table 1, one key implication of assuming constant effects is that the reduced-form quantities $\Delta Y_i/\Delta Z_j$, $\Delta Y_i/\Delta Z_i$, and $\Delta\bar{Y}/\Delta Z_i$ are identical across all agent pairs (i, j). This restriction allows us to derive testable implications of the classical linear-in-means model using OLS. In Lemma 1, we outline two such tests, which are straightforward to implement.

Lemma 1. Suppose that the linear-in-means model has a well-defined reduced form. Then:

(i) If $\beta_j = \beta_k$ in all groups, then for any $i \notin \{j, k\}$, the coefficient on Z_i in an OLS regression of Y_j on $(1, Z')'$ equals the coefficient on Z_i in an OLS regression of Y_k on $(1, Z')'$.

(ii) If Assumptions C.1 and C.4 hold, then the coefficient on $\bar{Z}_{-i}$ in an OLS regression of Y_i on $(1, Z_i, \bar{Z}_{-i})'$ equals the coefficient on Z_i in an OLS regression of $\bar{Y}_{-i}$ on $(1, Z_i, \bar{Z}_{-i})'$.

Part *(i)* of Lemma 1 provides a way to separately test Assumption C.1, which maintains that the interaction effects are fixed among all agents in the same group. A testable implication of this assumption is that the reduced form effects $\Delta Y_j/\Delta Z_i$ and $\Delta Y_k/\Delta Z_i$ are the same for any distinct agents $i, j, k \in \mathcal{N}$. In fact, for any agents i, j, and k, we show in the Appendix that $\beta_j = \beta_k$ if and only if $\Delta Y_j/\Delta Z_i = \Delta Y_k/\Delta Z_i$. The intuition behind this property is that, if two agents j and k are influenced in the same way by a third agent i, then any exogenous shock to agent i's outcome would produce identical spillover effects on agent j and agent k. Using this property, we can test whether $\beta_j = \beta_k$ by running OLS regressions of Y_j and Y_k on $(1, Z')'$ and checking whether the coefficient on any Z_i with $i \notin \{j, k\}$ differs between the two.

Part *(ii)* of Lemma 1 provides a way to jointly test Assumptions C.1 and C.4. Specifically, under these assumptions, the reduced-form effects $\Delta Y_i/\Delta\bar{Z}_{-i}$ and $\Delta\bar{Y}_{-i}/\Delta Z_i$ must be equal, since both correspond to the same homogeneous spillover effect, $\Delta Y_i/\Delta Z_j$. This equivalence

[6]See Kline & Tamer (2020) for a review. Bramoullé et al. (2009) formalize how to use network exclusions—where not all agents interact with one another—for identification of classical linear-in-means models.

imposes a testable restriction: in the population, the coefficient on $\bar{Z}_{-i}$ in a regression of Y_i on $(1, Z_i, \bar{Z}_{-i})'$ must equal the coefficient on Z_i in a regression of $\bar{Y}_{-i}$ on $(1, Z_i, \bar{Z}_{-i})'$. A difference between these coefficients would indicate that at least one of the assumptions is violated.

Frequently-Used IV Estimands

We now shift attention to a large class of IV estimands that use instruments to recover the interaction effect β. This quantity is often the main target parameter in constant effects models, and our framework nests a wide variety of existing approaches that are used to recover it.

We define an IV estimand for β that uses $\tilde{Z}_{-i}$ as the excluded instrument for $\bar{Y}_{-i}$ in an agent i's outcome equation. We allow $\tilde{Z}_{-i}$ to be any monotonic transformation of the vector Z_{-i}. In particular, we define $\tilde{Z}_{-i} = g(Z_{-i})$, where g is a monotone mapping taking values in the support of Z_{-i}.[7] Our specification encompasses a wide array of IV strategies, including: *(1)* using one instrument individually, *(2)* using multiple instruments jointly, and *(3)* using an increasing transformation of multiple instruments, e.g., a group-level average of $\{Z_j\}_{j \neq i}$. For any realization of z_i in the support of Z_i, we can write down an IV estimand as follows:

$$\beta_i^{\mathrm{IV}}(z_i) = \frac{\mathrm{Cov}(Y_i, \mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})|Z_i = z_i)}, \tag{8}$$

where $\mathbf{L}(\bar{Y}_{-i}|\tilde{Z}_{-i})$ represents the population fitted values from a regression of $\bar{Y}_{-i}$ on $(1, \tilde{Z}_{-i})$.

Under constant effects, the interaction effect β_i is point-identified from this IV estimand. In fact, even if the interaction effects vary within a group, the estimand would still recover β_i, provided that this interaction effect remains constant across groups. This result is well-established in the literature, and it is reviewed in textbooks tracing back to Fisher (1966).

Lemma 2. Suppose that the linear-in-means model has a well-defined reduced form. Then, if Assumption C.2 is satisfied, the IV estimand $\beta_i^{\mathrm{IV}}(z_i)$ will recover the interaction effect β_i.

This lemma provides us with a testable restriction for Assumption C.2. Specifically, under these assumptions, the IV estimand $\beta_i^{\mathrm{IV}}(z_i)$ always recovers the same parameter, regardless of which excluded instruments $\tilde{Z}_{-i}$ are used in the regression. Therefore, we can validate the classical linear-in-means assumptions by conducting an over-identification test. For $N > 2$, there may be multiple valid instruments $\{Z_j\}_{j \neq i}$ for the endogenous variable $\bar{Y}_{-i}$ in an agent i's outcome equation. We can leverage this over-identification to construct two IV estimands $\beta_i^{\mathrm{IV},1}$ and $\beta_i^{\mathrm{IV},2}$ for β_i using two distinct instruments $\tilde{Z}_{-i,1}$ and $\tilde{Z}_{-i,2}$, respectively. We can then empirically assess whether Assumptions C.2 holds by testing the null $H_0 : \beta_i^{\mathrm{IV},1} = \beta_i^{\mathrm{IV},2}$.

[7] Formally, we restrict g to the set of functions $\mathcal{G} = \{g : \mathrm{supp}(Z_{-i}) \to \mathbb{R}^n | g(z'_{-i}) \geq g(z_{-i}) \text{ for } z'_{-i} \geq z_{-i}\}$. For $\tilde{Z}_{-i}$ to be a relevant instrument, we require that g is strictly increases in at least one component of Z_{-i}.

3 Econometric Analysis under Heterogeneous Effects

We now relax Assumptions C.1-C.4 to allow agents to exhibit interaction effects of different signs and magnitudes, which vary both within and between groups. We treat $\alpha_g = [\alpha_{ig}]_{i \in \mathcal{N}_g}$, $\beta_g = [\beta_{ig}]_{i \in \mathcal{N}_g}$, and $\gamma_g = [\gamma'_{ig}]_{i \in \mathcal{N}_g}$ as random vectors jointly distributed according to a density f. We impose no parametric structure on f and allow for arbitrary dependence among the coefficients. For example, an agent i's interaction effect β_{ig} could be shaped by the interaction effects of i's peers, as well as by the interaction effects that are realized in the other groups. Moreover, since we permit the coefficients γ_{ig} to be heterogeneous, we allow for the possibility that the incidence of Z_{ig} varies and may even depend on the characteristics of other agents.[8]

3.1 Characterization of an Equilibrium

To analyze the equilibrium behavior of the linear-in-means model with heterogeneous interaction effects, we first derive the necessary and sufficient conditions for there to be a unique solution to the system of equations (1). The condition that we derive will significantly relax Assumption C.3. Specifically, rather than placing bounds on the signs and magnitudes of the endogenous interaction effects, our condition only rules out a single equality constraint.

Assumption I (Unique Solution). $\sum_{i \in \mathcal{N}_g} (1 - \beta_{ig}) \prod_{j \in \mathcal{N}_g \setminus i} (|\mathcal{N}_g| - 1 + \beta_{jg}) \neq 0$ for any group g.

Assumption I is a rank condition. It ensures that $I - \mathbf{B}_g$ is invertible, where I denotes the identity matrix and $\mathbf{B}_g$ is the adjacency matrix specifying the interaction effects in group g:

$$\mathbf{B}_g = \frac{1}{|\mathcal{N}_g| - 1} \begin{bmatrix} 0 & \beta_{1g} & \cdots & \beta_{1g} \\ \beta_{2g} & 0 & \cdots & \beta_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{|\mathcal{N}_g|g} & \beta_{|\mathcal{N}_g|g} & \cdots & 0 \end{bmatrix}. \tag{9}$$

This assumption rules out cases where the outcome equations (1) correspond to parallel lines. If these lines are parallel to each other, then they either never intersect or they overlap. In the first case, the model has no solution. In the second case, it has infinitely-many solutions.[9] By eliminating these two cases, Assumption I ensures that the equilibrium is well-defined.

We now present a closed form representation of the equilibrium, showing how the outcomes $\{Y_{ig}\}_{i \in \mathcal{N}_g}$ depend on the variables $\{Z_{ig}\}_{i \in \mathcal{N}_g}$ after accounting for spillover effects. In

[8] We allow α_g, β_g, and γ_g to be correlated with the group size and composition, as characterized by the set $\mathcal{N}_g$. This correlation could be economically meaningful. For example, the social pressures that individuals experience might depend on the number or types of peers within the group. Moreover, this correlation ties our hands by preventing us from using group size variation as a source of identification. Both Lee (2007) and Davezies et al. (2009) study how variation in group sizes can be used for identification of peer effects.

[9] Tamer (2003) discusses issues of *incoherency* and *incompleteness* of simultaneous equation models. When a model is incoherent, it has no solution. When a model is incomplete, it has multiple solutions. In our setting, nonintersecting lines makes the model incoherent, and overlapping lines makes the model incomplete. In the Appendix, we provide a graphical illustration of these two cases, discussing why they are both problematic.

general, spillovers have the potential amplify or suppress the impacts of $\{Z_{ig}\}_{i\in\mathcal{N}_g}$ on agents' outcomes. These distortions are driven by the interaction effects $\{\beta_{ig}\}_{i\in\mathcal{N}_g}$, which can be positive or negative in our framework. Moreover, as we allow the interaction effects to vary among agents, the nature of these distortions becomes more complex as the group size $|\mathcal{N}_g|$ grows larger. The following proposition gives a general characterization of the equilibrium.

Proposition 1. A unique solution to system (1) exists if and only if Assumption I holds. In equilibrium, the outcomes $\{Y_{ig}\}_{i\in\mathcal{N}_g}$ in group g satisfy $Y_{ig} = \alpha_{ig} + \beta_{ig}\bar{Y}_{g,-i} + Z'_{ig}\gamma_{ig}$, where:

$$\bar{Y}_g = \frac{\sum_{j\in\mathcal{N}_g}\left[\prod_{\ell\in\mathcal{N}_g\backslash j}\left(1+\frac{\beta_{\ell g}}{|\mathcal{N}_g|-1}\right)\right]\times(\alpha_{jg}+Z'_{jg}\gamma_{jg})}{|\mathcal{N}_g|\times\det(I-\mathbf{B}_g)}, \quad \text{and:}$$

$$\bar{Y}_{g,-i} = \frac{\sum_{j\in\mathcal{N}_g}\nu_{ijg}\times\left[\frac{\beta_{jg}}{|\mathcal{N}_g|-1}(\alpha_{ig}+Z'_{ig}\gamma_{ig})+(\alpha_{jg}+Z'_{jg}\gamma_{jg})\right]}{(|\mathcal{N}_g|-1)\times\det(I-\mathbf{B}_g)}, \quad \text{for } i\in\mathcal{N}_g.$$

Here, we define $\nu_{ijg} = 1$ for $|\mathcal{N}_g| = 2$ and $\nu_{ijg} = \prod_{\ell\in\mathcal{N}_g\backslash\{i,j\}}\left(1+\frac{\beta_{\ell g}}{|\mathcal{N}_g|-1}\right)$ for $|\mathcal{N}_g| > 2$. The determinant of $I-\mathbf{B}_g$ also has a closed-form expression, which is provided in the Appendix.

While prior work derives similar formulas for two- or three-agent special cases (e.g., Masten, 2017), our equilibrium formulas apply to groups of any size. Given this generality, our analysis extends to a wide range of settings with varying group size and composition.

Remark 1. Moment Determinacy.

Although Assumption I rules out models with parallel lines, it does not eliminate models with *nearly* parallel lines, in which $\det(I-\mathbf{B}_g)$ is close to zero with high probability. This distinction becomes important when we consider mean-based identification strategies, since the moments of the reduced form coefficients may not exist if $\det(I-\mathbf{B}_g)$ is very close to zero. For the reduced form moments to be well-defined, we need a slightly stronger assumption.

One sufficient condition for moment determinacy is that the vector of outcomes Y_g has a bounded support. Moreover, as Masten (2017) shows, the reduced form moments can exist even when Y_g takes full support if the tails of the outcome distributions are sufficiently thin. By reformulating Assumption A6 in Masten (2017) for our framework, we arrive at the following sufficient condition, which is expressed as a restriction on the structural parameters.[10]

Assumptions II (Sufficient Conditions for Moment Determinacy)**.**

II.1. $\mathrm{P}\left(\left|\sum_{i\in\mathcal{N}_g}(1-\beta_{ig})\prod_{j\in\mathcal{N}_g\backslash i}(|\mathcal{N}_g|-1+\beta_{jg})\right|\geq\tau\right)=1$ for some scalar $\tau>0$.

II.2. The marginal distributions of $\{\alpha_{ig}\}_{i,g}$ and $\{\gamma_{ig}\}_{i,g}$ have subexponential tails.

Remark 2. Preservation of Order.

Although not necessary for identification, it is often helpful for interpreting economic quan-

[10] For more discussion, as well as necessary conditions for moment determinacy, we refer to Masten (2017).

tities if the structural coefficients $\{\gamma_{ig}\}_{i,g}$ have the same signs (respectively) as the reduced form effects $\{\Delta Y_{ig}/\Delta Z_{ig}\}_{i,g}$. That is, if Z_{ig} has a positive direct effect on the outcome Y_{ig}, when does Z_{ig} have a positive effect on Y_{ig} in equilibrium? Consider the following condition.

Assumption III (Bounded Interactions)**.** $1 - |\mathcal{N}_g| < \beta_{ig} < 1$ for all agents i and groups g.

By ensuring that γ_{ig} and $\Delta Y_{ig}/\Delta Z_{ig}$ share the same sign, Assumption III rules out equilibrium behaviors that might seem illogical. For example, in a peer effects model, a student's achievement would not fall when the marginal utility of effort rises. In a household labor supply model, a person's income would not decrease after receiving a raise. In an oligopoly model, a firm's output would not fall as a consequence of becoming more productive.[11]

3.2 Defining Economic Quantities under Heterogeneous Effects

Next, we define and interpret the economic quantities in Table 1 under heterogeneous effects. As before, we ease notation by removing group subscripts and treating Z_i as one-dimensional.

<u>**Reduced Form Parameters**</u>

We first reexamine the reduced form parameters $\Delta Y_i/\Delta Z_j$, $\Delta Y_i/\Delta Z_i$, and $\Delta \bar{Y}/\Delta Z_i$, defined in equations (4)–(6). To aid in our analysis, we state the following corollary to Proposition 1.

Corollary 1. In equilibrium, the total effect of a unit increase in Y_j on Y_i, for $i \neq j$, equals:

$$\psi_{ij} = \beta_i \times \frac{\prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{N-1}\right)}{(N-1) \times \det(I - \mathbf{B})}.$$

Additionally, if Assumption III holds, then ψ_{ij} has the same sign as the interaction effect β_i.

This corollary expresses ψ_{ij} in terms of the individual interaction effects, providing a foundation for reinterpreting the reduced form parameters under heterogeneous effects. For example, the individual spillover effect is defined in equation (4) to be: $\Delta Y_i/\Delta Z_j = \gamma_j \times \psi_{ij}$. Under Assumption III, this effect has the same sign as $\gamma_j \times \beta_i$. Therefore, in equilibrium, a positive shock to Y_j would increase Y_i when β_i is positive and reduce Y_i when β_i is negative.

The total individual effect is defined in equation (5) as: $\Delta Y_i/\Delta Z_i = \gamma_i \times \big(1 + \frac{\beta_i}{N-1} \sum_{j \neq i} \psi_{ji}\big)$. This effect exceeds γ_i in magnitude whenever $\frac{\beta_i}{N-1} \sum_{j \neq i} \psi_{ji}$ is positive, which, by Corollary 1, occurs when all the interaction effects $\{\beta_i\}_{i=1}^N$ share the same sign. In such cases—whether the interaction effects are all positive (as in classical peer effects) or are all negative (as in

[11] Assumption III is a special case of Assumption I. Thus, it also ensures that there is a unique equilibrium. In addition, it implies that $\det(I - \mathbf{B}_g) > 0$ with probability 1 (see the Appendix for a proof). In a household labor supply model, Assumption III holds if all people value consumption: $\mu_{ig} \neq 1$ for all i. In an oligopoly model, it rules out Bertrand competition for firms with constant marginal costs: $(\theta_{ig}, \delta_{ig}) \neq (-1, 0)$ for all i. Such models do not possess an interior solution, since firms would always seek to undercut one another until they are all left with zero profit. This phenomenon is known as the Bertrand paradox (Edgeworth, 1925).

the household labor supply model or the model of firm oligopoly)—agents' actions are self-reinforcing, and spillovers amplify the impact of an exogenous shock on individual outcomes.

By Corollary 1, the total effect on the average, which is defined in equation (6), becomes:

$$\frac{\Delta \bar{Y}}{\Delta Z_i} = \frac{\prod_{\ell \neq i}\left(1 + \frac{\beta_\ell}{N-1}\right)}{N \times \det(I - \mathbf{B})} \times \gamma_i. \tag{10}$$

Under Assumption III, this quantity always has the same sign as the coefficient γ_i. Therefore, in a peer effects model, a policy that improves one student's performance always increases the average achievement level in the group. Similarly, in a household labor supply model, a wage boost for one individual always raises the total income of the household. In a model of firm oligopoly, improving one firm's productivity always increases overall market output.

Social Multiplier Effects

When the endogenous interaction effects are heterogeneous, Glaeser et al.'s (2003) social multiplier is not well-defined because $\bar{Z}$ could affect $\bar{Y}$ in different ways depending on which of the variables $\{Z_i\}_{i=1}^N$ is changed. In other words, the total effect of an exogenous shock on group outcomes depends on which agent(s) in the group are directly exposed to that shock. For heterogeneous effects, we can define an individual-specific social multiplier for an agent i.

$$M_{(i)}^{\text{heterog.}} = \frac{\sum_{j=1}^N \Delta Y_j / \Delta Z_i}{\Delta Y_i / \Delta Z_i} = \frac{1}{1 - \frac{1}{N-1}\sum_{j \neq i} \frac{\beta_j}{1 + \beta_j/(N-1)}}. \tag{11}$$

This quantity is defined as the ratio of the total effect of Z_i on $\sum_{j=1}^N Y_j$ to the individual effect of Z_i on Y_i. It generalizes the original definition of the social multiplier by accommodating heterogeneous effects. In a constant effects model, $M_{(i)}^{\text{heterog.}}$ reduces to M^{constant} for every i. Additionally, as the size of the group N becomes large, $M_{(i)}^{\text{heterog.}}$ tends to $(1 - \frac{1}{N-1}\sum_{j \neq i} \beta_j)^{-1}$.

The notion of an individual-specific social multiplier is particularly intuitive when group members assume different roles. In the household labor supply example, $M_{(i)}^{\text{heterog.}}$ measures how an exogenous change in person i's wage would affect total household income relative to i's individual income Y_i. If there is only one primary earner in the household, then it is likely that these multipliers differ across household members i. For example, in a two-person household, $M_{(i)}^{\text{heterog.}}$ equals $1 - \mu_j$, which captures the second household member j's trade-off between consumption and leisure. If member j places high value on leisure (so μ_j is large), then j is more willing to work less when i earns more. In this case, the multiplier $M_{(i)}^{\text{heterog.}}$ is small since the total impact of raising i's wage on total household income would be heavily offset by a reduction in j's labor supply. Alternatively, if j places high value on consumption, then his/her labor supply is less responsive to i's wage, and the multiplier $M_{(i)}^{\text{heterog.}}$ is large.[12]

[12] Note that the multiplier effects are always less than one in this example, since strategic substitutability suppresses the impact of exogenous wage shocks on total household income, which ensures that $M_{(i)}^{\text{heterog.}} < 1$.

Averaging across agents, we can construct an aggregate social multiplier effect $M^{\text{heterog.}}$, equal to $\frac{1}{N}\sum_{i=1}^{N} M_{(i)}^{\text{heterog.}}$. Alternatively, we can take $M^{\text{heterog.}}$ as the ratio of average effects:

$$M^{\text{heterog.}} = \frac{\sum_{i=1}^{N} \Delta\bar{Y}/\Delta Z_i}{\frac{1}{N}\sum_{j=1}^{N} \Delta Y_j/\Delta Z_j} = \frac{\frac{1}{N}\sum_{j=1}^{N}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\gamma_j}{\frac{1}{N}\sum_{j=1}^{N}\left(1+\frac{\beta_j}{N-1}\right)^{-1}\left(1-\frac{1}{N-1}\sum_{k\neq j}\frac{\beta_k}{1+\beta_k/(N-1)}\right)\gamma_j}. \quad (12)$$

While they have slightly different interpretations, both versions of the aggregate social multiplier reduce to the original definition under constant effects. Throughout the rest of the paper, we take the expression in (12) as our definition of the aggregate social multiplier. If γ_i is constant across agents i, then $M^{\text{heterog.}}$ tends to $\left(1-\frac{1}{N}\sum_{i=1}^{N}\beta_i\right)^{-1}$ as N grows large.

3.3 Analysis of OLS and IV under Heterogeneous Effects

We now analyze what can and cannot be learned from frequently used OLS and IV estimands for linear-in-means models under heterogeneous effects. We show that, while these estimands do not lead to point identification, they still carry information about key economic quantities.

To accommodate heterogeneous effects, we replace the instrument exogeneity condition with the assumption that $Z_g \perp (\alpha_g, \beta_g, \gamma_g, \mathcal{N}_g)$. This assumption is standard in the literature on random coefficients. It ensures that the unobserved parameters are statistically independent of the vector of observables Z_g.[13] As in Section II, we ease notation by omitting group subscripts and treating Z_{ig} as one-dimensional. We also assume that Z_{ig} is an individual-level shifter, and that the shifters are not perfectly collinear. To ensure instrument relevance, we assume that $\gamma_{ig} \neq 0$ for every i and g, so that Z_{ig} has a nonzero effect on observed outcomes.[14]

3.3.1 Empirical Analysis of OLS Estimands

We first analyze the OLS estimands $\beta^{\text{OLS}}(Y_i)$ and $\beta^{\text{OLS}}(\bar{Y})$ that are defined in Table 2. Under heterogeneous effects, these estimands recover the average reduced form effects across groups.

Proposition 2. In a linear-in-means model with heterogeneous effects, $\beta^{\text{OLS}}(Y_i)$ and $\beta^{\text{OLS}}(\bar{Y})$ recover the average reduced form effects $\mathrm{E}(\Delta Y_i/\Delta Z_i)$, $\{\mathrm{E}(\Delta Y_i/\Delta Z_j)\}_{j\neq i}$, and $\{\mathrm{E}(\Delta\bar{Y}/\Delta Z_j)\}_j$.

This proposition reveals that, even under heterogeneous effects, the OLS estimands $\beta^{\text{OLS}}(Y_i)$ and $\beta^{\text{OLS}}(\bar{Y})$ offer insight into how individual-level shocks affect equilibrium outcomes. In a peer effects setting, they capture average equilibrium responses (across classrooms) of student achievement to policy interventions. In a household labor supply context, they reflect average responses (across families) of earnings to individual wage shocks. In an oligopoly setting, they measure average responses (across markets) of firm output to firm-specific cost shocks.

[13] If Z_g includes covariates, then we can relax Assumption I to allow for independence of individual-level shifters conditional on covariates: $Z_g^s \perp (\alpha_g, \beta_g, \gamma_g, \mathcal{N}_g)|Z_g^c$, where Z_g^s are shifters and Z_g^c are covariates. In addition, if the set of agents $\mathcal{N}_g$ in a group is observed, then we can relax it by writing $Z_g \perp (\alpha_g, \beta_g, \gamma_g)|\mathcal{N}_g$.

[14] This assumption can be relaxed to allow for instruments that affect the outcomes for a subset of agents.

For the OLS estimands to be well-specified in the presence of within-group heterogeneity, it is essential to include all individual shifters $\{Z_j\}_{j=1}^N$ as separate regressors. A researcher might be tempted to simplify these regressions by instead computing the following estimands:

$$\check{\beta}^{\text{OLS}}(Y_i) = \mathrm{E}(\check{Z}_i \check{Z}_i')^{-1}\,\mathrm{E}(\check{Z}_i Y_i) \quad \text{and} \quad \check{\beta}^{\text{OLS}}(\bar{Y}) = \mathrm{E}(\check{Z}\check{Z}')^{-1}\,\mathrm{E}(\check{Z}\bar{Y}),$$

corresponding to OLS regressions of Y_i on $\check{Z}_i = (1, Z_i, \bar{Z}_{-i})$ and of $\bar{Y}$ on $\check{Z} = (1, \bar{Z})$. These estimands successfully recover average reduced form effects under Assumptions C.1 and C.4, where β_i and γ_i are homogeneous within each group. However, when β_i and γ_i vary among agents in a group, these simplified regressions are no longer valid. The reason is that within-group heterogeneity causes $\Delta Y_i/\Delta Z_j$ and $\Delta \bar{Y}/\Delta Z_j$ to vary across agents j. Therefore, any OLS regression that includes averages of Z, while excluding $\{Z_j\}_{j=1}^N$ as individual regressors, suffers from omitted variable bias.[15] This bias arises even with constant effects across groups. Indeed, as long as β_i and γ_i differ within a group, the simplified regressions are misspecified.

OLS Estimands for Social Multiplier Effects

For constant effects models, the social multiplier $M^{\text{constant}} = (\Delta\bar{Y}/\Delta\bar{Z})/(\Delta Y_i/\Delta Z_i)$ is point identified from OLS estimands. Specifically, $\Delta\bar{Y}/\Delta\bar{Z}$ is recovered from regressing $\bar{Y}$ on $(1, \bar{Z})$ and $\Delta Y_i/\Delta Z_i$ is recovered from regressing Y_i on $(1, Z_i, \bar{Z}_{-i})$. However, in the linear-in-means model with heterogeneous effects, the social multiplier is no longer point identified from OLS.

To understand why OLS estimands do not recover social multipliers under heterogeneous effects, first recall that M^{constant} is not well-defined in the case of within-group heterogeneity. Instead, we define individual-specific multipliers $M_{(i)}^{\text{heterog.}}$ and aggregate multipliers $M^{\text{heterog.}}$, which are better suited for settings where agents in a group face different interaction effects. If the interactions are constant across groups, then $M_{(i)}^{\text{heterog.}}$ and $M^{\text{heterog.}}$ are both identified from correctly specified OLS regressions, following the previous discussion. However, if the interaction effects vary across groups, then these regressions instead recover the estimands:

$$M_{(i)}^{\text{OLS}} = \frac{\sum_{j=1}^N \mathrm{E}(\Delta Y_j/\Delta Z_i)}{\mathrm{E}(\Delta Y_i/\Delta Z_i)} \quad \text{and} \quad M^{\text{OLS}} = \frac{\sum_{i=1}^N \mathrm{E}(\Delta\bar{Y}/\Delta Z_i)}{\frac{1}{N}\sum_{j=1}^N \mathrm{E}(\Delta Y_j/\Delta Z_j)}.$$

These estimands represent ratios of average equilibrium effects across groups. Yet, they do not correspond to the economic quantities of interest in Table 1. As we show in Section V, we may still be able to use OLS to place informative bounds on the social multiplier effects.

3.3.2 Empirical Analysis of IV Estimands

We now reexamine the IV estimand, which is defined in equation (8). Under heterogeneous effects, IV does not lead to point identification of β_i. This negative result motivates our subsequent analysis, examining: When is the IV estimand informative about interaction effects?

[15] If $\{Z_j\}_{j=1}^N$ are all uncorrelated, then the coefficient on Z_i in a regression of Y_i on $(1, Z_i, \bar{Z}_{-i})$ would still recover the average total individual effect $\mathrm{E}(\Delta Y_i/\Delta Z_i)$. Yet, the other coefficients are biased by construction.

We establish conditions under which the IV estimand $\beta_i^{\mathrm{IV}}(z_i)$ will be a positively-weighted average of β_i, which is a minimal requirement for it to be informative about social interaction effects. A standard condition for this property, which is widely used in the treatment effects literature, is proposed by Imbens & Angrist (1994). It requires that the endogenous variable $\bar{Y}_{-i}$ is affected uniformly by any change in the instrument $\tilde{Z}_{-i}$. If we take $\tilde{Z}_{-i}$ to be Z_{-i} (or if $\tilde{Z}_{-i}$ is a one-to-one function of Z_{-i}) then this condition has the following characterization.

Assumption IAM *(Imbens-Angrist Monotonicity)***.** For any vectors (z_{-i}, z_i) and (z'_{-i}, z_i) in the support of Z, either $\mathrm{P}\left(\bar{Y}_{-i}(z_{-i}, z_i) \geq \bar{Y}_{-i}(z'_{-i}, z_i)\right) = 1$ or $\mathrm{P}\left(\bar{Y}_{-i}(z_{-i}, z_i) \leq \bar{Y}_{-i}(z'_{-i}, z_i)\right) = 1$.

We argue that this condition is plausible in settings where the interactions take place between two agents, but we demonstrate that it is unlikely to hold with groups of three or more agents.

Pairs of Agents $(N = 2)$

We consider a special case of the model where the interactions take place between two agents.

$$Y_1 = \alpha_1 + \beta_1 Y_2 + \gamma_1 Z_1 \tag{13}$$

$$Y_2 = \alpha_2 + \beta_2 Y_1 + \gamma_2 Z_2. \tag{14}$$

This special case allows us to study peer effects between pairs of students, joint labor supply decisions in two-person households, and the strategic interactions among firms in duopolies.

For any $j \neq i$, the IV estimand equals $\beta_i^{IV}(z_i) = \mathrm{Cov}(Y_i, Z_j | Z_i = z_i)/\,\mathrm{Cov}(Y_j, Z_j | Z_i = z_i)$. This estimand can be expressed as a weighted average of all the potential realizations of β_i.

$$\beta_i^{IV}(z_i) = \int_{\mathrm{supp}(\beta_i)} b_i \times \omega(b_i) db_i, \quad \text{where} \quad \omega(b_i) = \frac{\mathrm{E}(\Delta Y_j / \Delta Z_j | \beta_i = b_i) f_{\beta_i}(b_i)}{\mathrm{E}(\Delta Y_j / \Delta Z_j)}. \tag{15}$$

Observe that larger weights $\omega(b_i)$ are placed on values of β_i in groups where the outcome Y_j is more responsive to the instrument Z_j. For the weights to be non-negative, we can impose IAM monotonicity, which requires that Y_j is uniformly affected in the same direction by Z_j. This condition holds if and only if the coefficient γ_j retains the same sign across all networks:

$$\mathrm{P}(\gamma_j \geq 0) = 1 \text{ or } \mathrm{P}(\gamma_j \leq 0) = 1. \tag{16}$$

This condition does not impose restrictions on the interaction effects (β_1, β_2) in the model.[16]

Example (Peer Effects). Consider a model of peer effects with two students: i and j. Let Z_j indicate whether student j receives a scholarship, and assume that this scholarship always raises student achievement, such that $\mathrm{P}(\gamma_j \geq 0) = 1$. In this case, IV recovers the average peer effect β_i in groups where student j's achievement is most impacted by the scholarship.

Example (Household Labor Supply). Suppose that each household has two members, i

[16] An alternative sufficient condition is: $\gamma_j \perp (\beta_1, \beta_2)$. However, this condition does not extend to $N > 2$.

and j, and let Z_j be a policy that increases person j's wage. Then, IV measures the average second earner effect μ_i in households where j's income is particularly affected by the policy.

Example (Duopoly). Consider a duopoly, and let Z_j be a technological shock that always raises the productivity of firm j, i.e., $\mathrm{P}(\lambda_{j1} \leq 0) = 1$. In this case, IV would estimate the average conduct parameter for firm i in the markets where j is most responsive to the shock.

Groups of Three Agents ($N = 3$)

For peer groups of more than two agents, the IAM assumption is more restrictive with respect to the interaction effects. To unpack these restrictions, we will examine the three-agent case.

$$Y_1 = \alpha_1 + \beta_1\Big(\frac{Y_2 + Y_3}{2}\Big) + \gamma_1 Z_1 \tag{17}$$

$$Y_2 = \alpha_2 + \beta_2\Big(\frac{Y_1 + Y_3}{2}\Big) + \gamma_2 Z_2 \tag{18}$$

$$Y_3 = \alpha_3 + \beta_3\Big(\frac{Y_1 + Y_2}{2}\Big) + \gamma_3 Z_2. \tag{19}$$

For distinct agents $i, j, k \in \{1, 2, 3\}$, the endogenous variable in agent i's outcome equation is $\bar{Y}_{-i} = \frac{1}{2}(Y_j + Y_k)$. A researcher can use either Z_j or Z_k as a valid instrument for $\bar{Y}_{-i}$. In this example, we focus on an IV strategy that uses both instruments jointly, i.e., $\tilde{Z}_{-i} = (Z_j, Z_k)$. As in the two-agent case, we can interpret $\beta_i^{\mathrm{IV}}(z_i)$ as a weighted average of interaction effects, where larger weights are given to values of β_i in groups where $\bar{Y}_{-i}$ is more affected by Z_{-i}.

If we impose IAM, then $\beta_i^{IV}(z_i)$ will be a positively-weighted average of β_i's. However, as shown in Figure 1, IAM places strong conditions on the reduced form effects $\Delta\bar{Y}_{-i}/\Delta Z_j$ and $\Delta\bar{Y}_{-i}/\Delta Z_k$.[17] For binary instruments, it requires that these effects have the same signs in all networks and that one of these effects is always larger in magnitude than the other one. For continuous instruments, it requires that the ratio of $\Delta\bar{Y}_{-i}/\Delta Z_j$ to $\Delta\bar{Y}_{-i}/\Delta Z_k$ is constant.

The restrictions on the reduced form also impose restrictions on the interaction effects.

Lemma 3. When $N = 3$ and (Z_j, Z_k) are binary, Assumption IAM holds if and only if:

(*i*) $\mathrm{P}(\gamma_\ell \geq 0) = 1$ or $\mathrm{P}(\gamma_\ell \leq 0) = 1$, for $\ell \in \{j, k\}$.

(*ii*) $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \geq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$ or $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \leq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$.

Example (Peer Effects). Suppose that Z_j and Z_k are binary variables indicating whether students j and k, respectively, receive a scholarship. For simplicity, assume that γ_j and γ_k are uniform within and across peer groups. Then, IAM requires that one student always has a larger interaction effect than the other student: either $\mathrm{P}(\beta_j \geq \beta_k) = 1$ or $\mathrm{P}(\beta_j \leq \beta_k) = 1$.[18]

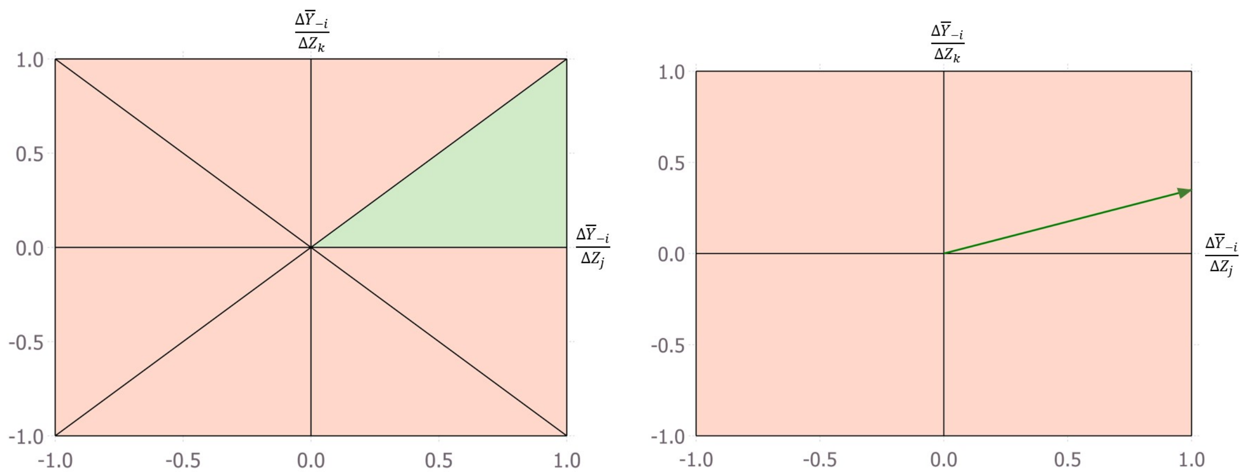
[17] Specifically, the IAM assumption imposes a total order on a vector space, requiring that the relation $\succeq$, where $z_{-i} \succeq z'_{-i}$ if and only if $\mathrm{P}\left(\bar{Y}_{-i}(z_{-i}, z_i) \geq \bar{Y}_{-i}(z'_{-i}, z_i)\right) = 1$, is a total order on the support of Z_{-i}.

[18] If the indices j and k are chosen arbitrarily, then one could overcome this restriction by defining j to be the member of the peer group who experiences the most social pressure. However, if j and k take on specific

Example (Household Labor Supply). Suppose that Z_j and Z_k are binary factors influencing the wages of household members j and k, respectively. In this case, IAM requires that one person always values leisure more than the other: either $\mathrm{P}(\mu_j \geq \mu_k) = 1$ or $\mathrm{P}(\mu_j \leq \mu_k) = 1$.

Example (Oligopoly). Suppose that Z_j and Z_k are binary productivity shocks to firms j and k, respectively. If the coefficients λ_{j1} and λ_{k1} are constant within and across markets, then IAM implies that $\delta_j + b\theta_j$ is always greater than (or always less than) $\delta_k + b\theta_k$. To interpret this statement, recall that δ_j and δ_k are the slopes of firms' marginal cost curves, and $b\theta_j$ and $b\theta_k$ are the (conjectured) indirect effects of firms' actions on the market price. Unless the indices j and k are chosen to satisfy this restriction, it is hard to justify in practice.

Figure 1. Illustration of IAM Conditions for Two Instruments



Case 1. Binary Instruments (Z_j, Z_k) *Case 2.* Continuous Instruments (Z_j, Z_k)

Notes. These plots display feasible regions of the vector $(\Delta\bar{Y}_{-i}/\Delta Z_j, \Delta\bar{Y}_{-i}/\Delta Z_k)$ under Assumption IAM.

If the instruments Z_j and Z_k are continuous, then IAM imposes even stronger restrictions.

Lemma 4. When $N = 3$ and (Z_j, Z_k) are continuous, Assumption IAM holds if and only if:

(*i*) $\mathrm{P}(\gamma_\ell \geq 0) = 1$ or $\mathrm{P}(\gamma_\ell \leq 0) = 1$, for $\ell \in \{j, k\}$.

(*ii*) $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \geq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$ or $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \leq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$.

Examples. For the peer effects example where $\gamma_j = \gamma_k$, Assumption IAM requires that β_j is a deterministic linear function of β_k, such that $\beta_j = 2(a-1) + a\beta_k$ for $a \in \mathbb{R}$. For a household labor supply model where the wages (W_j, W_k) are used as instruments, this assumption requires that household member j and k's preferences over leisure and consumption are deterministic functions of one another, where $\frac{2-\mu_j}{1-\mu_j} = a \times \frac{2-\mu_k}{1-\mu_k}$. Finally, for an oligopoly model with $\lambda_{j1} = \lambda_{k1}$, it implies that $(\delta_j + b\theta_j) = a \times (\delta_k + b\theta_k) + 1.5b(a-1)$. We are not aware

roles, such as "teacher and student" or "parent and child", then this relabeling approach will not be feasible.

of any meaningful justification for these restrictions. So, for any model with heterogeneous effects and two continuous instruments, IAM would be particularly difficult to rationalize.[19]

<u>Alternative Conditions for Positive Weights</u>

In order to overcome the economic restrictions implied by Imbens-Angrist monotonicity, we propose an alternative assumption, which is sufficient for the IV estimand to be a positively-weighted average of interaction effects. Specifically, under a testable condition on the instrument correlation structure, we can relax IAM by imposing a weaker form of monotonicity.

Assumption PM (Partial Monotonicity)**.** For any $j \neq i$ and any (z_j, z_{-j}) and (z'_j, z_{-j}) in the support of Z, either $\mathrm{P}\left(\bar{Y}_{-i}(z_j, z_{-j}) \geq \bar{Y}_{-i}(z'_j, z_{-j})\right) = 1$ or $\mathrm{P}\left(\bar{Y}_{-i}(z_j, z_{-j}) \leq \bar{Y}_{-i}(z'_j, z_{-j})\right) = 1$.

This form of monotonicity is studied by Mogstad et al. (2021) as an alternative to the Imbens-Angrist condition. It requires that monotonicity holds separately for each instrument instead of for the entire instrument vector. If there is only one instrument, then both assumptions are the same. If there are multiple instruments, then PM is weaker than IAM.

To see what PM implies about the structural parameters, consider the following lemma.

Lemma 5. Assumption PM holds if and only if $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$ for all $j \neq i$.

This result is perhaps surprising given the complex nature of the model. It reveals that PM imposes no restrictions on the interaction effects. Instead, it only requires that the random coefficient γ_j on each instrument Z_j, where $j \neq i$, retains the same sign across all groups.

We now introduce a testable condition that restricts the correlation structure of Z. This condition places a bound the covariances of the instruments $\{Z_j\}_{j \neq i}$ in relation to the average reduced form effects $\{\mathrm{E}(\Delta \bar{Y}_{-i}/\Delta Z_j)\}_{j \neq i}$, which are point identified from OLS regressions.

Assumption NNW *(No Negative Weights)***.** Fix some $z_i \in \operatorname{supp}(Z_i)$. For any $j, k \in \mathcal{N} \setminus i$:

$$\operatorname{Cov}(Z_j, Z_k|z_i) \notin \left(-\sum_{\ell \notin \{i,j\}} \frac{\mathrm{E}(\Delta \bar{Y}_{-i}/\Delta Z_\ell)}{\mathrm{E}(\Delta \bar{Y}_{-i}/\Delta Z_j)} \operatorname{Cov}(Z_\ell, Z_k|z_i), -\sum_{\ell \notin \{i,k\}} \frac{\mathrm{E}(\Delta \bar{Y}_{-i}/\Delta Z_\ell)}{\mathrm{E}(\Delta \bar{Y}_{-i}/\Delta Z_k)} \operatorname{Cov}(Z_\ell, Z_j|z_i) \right).$$

This assumption holds if all the instruments Z_{-i} are uncorrelated. Also, if the components of γ_{-i} share the same sign, then it holds when no two instruments are negatively correlated.

Lemma 6. Assumption NNW is satisfied if either: *(1)* $\operatorname{Cov}(Z_j, Z_k|z_i) = 0$ for all $j, k \in \mathcal{N} \setminus i$ or if *(2)* both $\operatorname{Cov}(Z_j, Z_k|z_i) \geq 0$ for all $j, k \in \mathcal{N} \setminus i$ and $\mathrm{P}(\gamma_{-i} \geq 0) = 1$ or $\mathrm{P}(\gamma_{-i} \leq 0) = 1$.

Note that NNW can be tested empirically as the terms in this condition are identified in the data. So, one can assess whether this restriction holds without making economic arguments.

[19]Even if the instrument $\tilde{Z}_{-i}$ is a non-invertible function of (Z_j, Z_k), Assumption IAM is often still highly restrictive. For example, in the case where $\tilde{Z}_{-i}$ is a linear combination of Z_j and Z_k, the restrictions implied by Lemmas 3 and 4 are similar, if not unchanged. Moreover, even if we were to use only one instrument, setting $\tilde{Z}_{-i} = Z_j$, the restrictions on the interaction effects do not go away unless Z_j and Z_k are uncorrelated.

Suppose that we use a combination of the variables in Z_{-i} as our excluded instrument. Then PM and NNW ensure that the estimand $\beta_i^{\mathrm{IV}}(z_i)$ is a positively-weighted average of β_i's.

Proposition 3. Choose $\tilde{Z}_{-i} \subseteq Z_{-i}$, and suppose that Assumptions PM and NNW both hold. Then the IV estimand is a positively-weighted average of instrument-specific IV estimands:

$$\beta_i^{\mathrm{IV}}(z_i) = \sum_{j \neq i} \omega_j \times \frac{\mathrm{Cov}(Y_i, Z_j | z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, Z_j | z_i)}, \quad \text{where: } \sum_{j \neq i} w_j = 1 \text{ and } w_j \geq 0, \ \forall j \neq i.$$

Additionally, the IV estimand represents a positively-weighted average of interaction effects:

$$\beta_i^{\mathrm{IV}}(z_i) = \int_{\mathrm{supp}(\beta_i)} \beta_i \times \omega(\beta_i | z_i) d\beta_i, \quad \text{where: } \int \omega(\beta_i | z_i) d\beta_i = 1 \text{ and } \omega(\beta_i | z_i) \geq 0, \ \forall \beta_i.$$

From this proposition, we also derive a corollary that applies for any type of instrument $\tilde{Z}_{-i}$.

Corollary 2. For any choice of $\tilde{Z}_{-i}$, the IV estimand is a positively-weighted average of β_i if:

(i) $\mathrm{P}(\gamma_{-i} \geq 0) = 1$ or $\mathrm{P}(\gamma_{-i} \leq 0) = 1$.

(ii) $\mathrm{corr}(Z_j, Z_k | z_i) \geq 0$, for any $j, k \neq i$.

To interpret these results, we now reconsider the special case of the model with three agents.

Groups of Three Agents ($N = 3$)

When there are three agents $i, j, k \in \{1, 2, 3\}$, Assumption PM requires that γ_j and γ_k retain the same signs across all peer groups, and Assumption NNW simplifies in the following way:

$$\mathrm{Cov}(Z_j, Z_k | z_i) \notin \left(-\frac{\mathrm{E}(\Delta \bar{Y}_{-i} / \Delta Z_j)}{\mathrm{E}(\Delta \bar{Y}_{-i} / \Delta Z_k)} \mathrm{Var}(Z_j | z_i), -\frac{\mathrm{E}(\Delta \bar{Y}_{-i} / \Delta Z_k)}{\mathrm{E}(\Delta \bar{Y}_{-i} / \Delta Z_j)} \mathrm{Var}(Z_k | z_i) \right). \tag{20}$$

Example (Peer Effects). First, consider a peer effects model where Z_j and Z_k are factors that raise the achievement of students j and k, respectively. If these factors are not negatively correlated, then the IV estimand $\beta_i^{\mathrm{IV}}(z_i)$ is a causal parameter. It measures the average peer effect β_i in groups where the mean performance of students j and k is most affected by $\tilde{Z}_{-i}$.

Example (Household Labor Supply). Suppose that Z_j and Z_k are the wages of household members j and k, respectively. If these wages are not negatively correlated, then $\beta_i^{\mathrm{IV}}(z_i)$ represents the average value of μ_i in households where the earnings of j and k are most improved by $\tilde{Z}_{-i}$, i.e., where j and k are least inclined to reduce their labor when wages rise.

Example (Oligopoly). Suppose that Z_j and Z_k are positive productivity shocks that are experienced by firms j and k, respectively. As long as these two shocks are not negatively correlated, the parameter $\beta_i^{\mathrm{IV}}(z_i)$ is causal. It measures the average conduct parameter of firm i in markets where the mean output of firms j and k is most responsive to the instrument $\tilde{Z}_{-i}$.

Conditional IV Estimation Using One Instrument

In cases where Assumptions NNW and PM fail to hold, an alternative IV specification may still recover a positively-weighted average of interaction effects. Consider the IV estimand:

$$\beta_i^{IV}(z_{-j}) = \frac{\mathrm{Cov}(Y_i, Z_j | Z_{-j} = z_{-j})}{\mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_{-j} = z_{-j})} \tag{21}$$

This estimand uses only one instrument Z_j, while controlling for all other instruments Z_{-j}. To interpret this estimand as a positively-weighted average of the interaction effects, we only require that $\Delta\bar{Y}_{-i}/\Delta Z_j$ has the same sign across all networks. This monotonicity condition imposes the same parametric restriction as in the $N = 2$ case. In particular, $\beta_i^{IV}(z_{-j})$ equals a positively-weighted average of β_i-values if and only if $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$.[20] Higher weights are put on β_i-values in groups where $\bar{Y}_{-i}$ is more affected by the instrument Z_j.

3.4 Learning about Interaction Effects and Multipliers under Heterogeneous Effects

In this section, we show how to use OLS and IV regressions to learn about endogenous interaction effects and social multipliers in the linear-in-means model with heterogeneous effects.

3.4.1 Using IV to Bound Average Interaction Effects

First, we show how the IV estimand compares to an unweighted average of interaction effects. The following proposition demonstrates that this relationship is governed by $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$, which is defined in Corollary 1. This parameter has an important economic interpretation: it determines how an individual i's outcome Y_i affects the average outcome $\bar{Y}_{-i}$ of i's peers.

Proposition 4. Let $\beta_i^{\mathrm{IV}}(z_i)$ be a positively-weighted average of β_i and $\mathrm{E}(\beta_i|\beta_{-i}, \gamma_{-i}) = \mathrm{E}(\beta_i)$.

(*i*) If $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji} > 0$ with probability 1, then $\beta_i^{\mathrm{IV}}(z_i) > \mathrm{E}(\beta_i)$.

(*ii*) If $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji} < 0$ with probability 1, then $\beta_i^{\mathrm{IV}}(z_i) < \mathrm{E}(\beta_i)$.

There are notable examples where the sign of $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$ can be easily determined. Under Assumption III, $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$ is positive if $\beta_j > 0$ for all j and negative if $\beta_j < 0$ for all j.[21]

Examples. Suppose that all the interaction effects share the same sign. Then the IV estimand overstates the magnitude of $\mathrm{E}(\beta_i)$ for any agent i. Namely, for a peer effects model with positive social interactions, IV would overestimate the average peer effect. For a household labor supply model, it would overestimate the average added earner effect. Finally, for an oligopoly model, it would overestimate the average conduct parameter in the market.

[20] Averaging over Z_{-j}, we can also define the following IV estimand $\beta_i^{IV} = \int \beta_i^{IV}(z_{-j}) f_{Z_{-j}}(z_{-j}) dz_{-j}$.

[21] In the Appendix, we show how this result extends to cases where β_i and β_{-i} are statistically dependent.

Remark. While these examples may suggest that IV generally overestimates the magnitude of $\mathrm{E}(\beta_i)$, there are also notable exceptions. For example, consider a peer effects model where $\beta_i < 0$ and $\beta_j > 0$ for every $j \neq i$. In this setting, everyone seeks to conform to the average action in the group, except for person i, who wishes to deviate. Since $\frac{1}{N-1}\sum_{j\neq i}\psi_{ji}$ is below zero in this case, the IV estimand $\beta_i^{\mathrm{IV}}(z_i)$ would understate the magnitude of $\mathrm{E}(\beta_i)$.

Pairs of Agents ($N = 2$)

For two-agent groups, we draw comparisons to the mean with the following decomposition.

$$\beta_i^{IV}(z_i) = \mathrm{E}(\beta_i) + \frac{\mathrm{Cov}[\beta_i, \gamma_j/(1-\beta_1\beta_2)]}{\mathrm{E}[\gamma_j/(1-\beta_1\beta_2)]}. \tag{22}$$

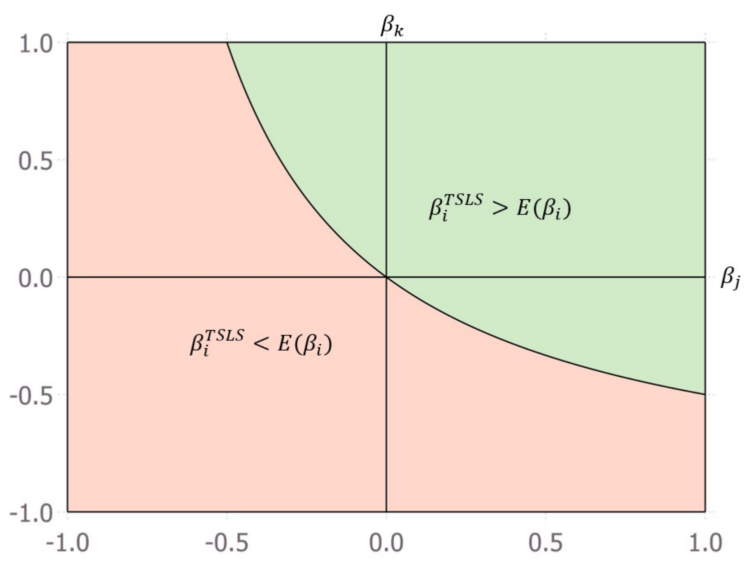
If β_i is mean independent of (β_j, γ_j), then the relationship between $\beta_i^{IV}(z_i)$ and $\mathrm{E}(\beta_i)$ is fully governed by agent j's interaction effect β_j. In particular, *(i)* and *(ii)* in Proposition 4 become:

(i) If $\beta_j > 0$ with probability 1, then $\beta_i^{IV}(z_i) > \mathrm{E}(\beta_i)$.

(ii) If $\beta_j < 0$ with probability 1, then $\beta_i^{IV}(z_i) < \mathrm{E}(\beta_i)$.

One implication of Proposition 4 is that, if β_1 and β_2 have the same sign within and across groups, then IV necessarily overstates the magnitudes of $\mathrm{E}(\beta_1)$ and $\mathrm{E}(\beta_2)$. Alternatively, if β_1 and β_2 always have opposite signs, then IV understates the magnitudes of $\mathrm{E}(\beta_1)$ and $\mathrm{E}(\beta_2)$.

Figure 2. Cases Where $\beta_i^{IV}(z_i) > \mathrm{E}(\beta_i)$ for Three-Agent Groups



Notes. This figure depicts values of (β_j, β_k) where the IV estimand overstates the average interaction effect.

Groups of Three Agents ($N = 3$)

Suppose that each group contains three agents. Then, *(i)* and *(ii)* in Theorem 2 reduce to:

(i) If $\beta_j + \beta_k + \beta_j\beta_k > 0$ with probability 1, then $\beta_i^{\text{IV}}(z_i) > \text{E}(\beta_i)$.

(ii) If $\beta_j + \beta_k + \beta_j\beta_k < 0$ with probability 1, then $\beta_i^{\text{IV}}(z_i) < \text{E}(\beta_i)$.

In Figure 2, we plot the settings where the sum $\beta_j+\beta_k+\beta_j\beta_k$ is positive. If β_j and β_k share the same sign, then the relationship between $\beta_i^{\text{IV}}(z_i)$ and $\text{E}(\beta_i)$ is unambiguous. Alternatively, if these interaction effects have different signs, then it is harder to compare $\beta_i^{\text{IV}}(z_i)$ with $\text{E}(\beta_i)$.

3.4.2 Using OLS to Test for Endogenous Interaction Effects and Multipliers

We now demonstrate how to use OLS regressions to test for the presence of social multipliers and endogenous interaction effects, as well as to learn about the signs and magnitudes of these interaction effects under heterogeneous effects. Our tests will utilize the average equilibrium quantities $\{\text{E}(\Delta Y_j/\Delta Z_i)\}_{i,j}$, $\{\text{E}(\Delta \bar{Y}/\Delta Z_i)\}_i$, and $\{\text{E}(\Delta \bar{Y}_{-i}/\Delta Z_i)\}_i$, all of which are are point identified from correctly specified OLS regressions, following the discussion in Section IV.A.

Before presenting our tests, we first establish the following proposition, which shows how endogenous interaction effects and social multipliers relate to various reduced form quantities.

Proposition 5. Let $\gamma_i > 0$ for all i. Then, under Assumptions I, II, and III, it follows that:

(a) $M_{(i)}^{\text{heterog.}} - 1$ has the same sign as $\Delta \bar{Y}_{-i}/\Delta Z_i$.

(b) $M^{\text{heterog.}} - 1$ has the same sign as $\sum_{i=1}^{N} \Delta \bar{Y}_{-i}/\Delta Z_i$.

(c) β_j has the same sign as $\Delta Y_j/\Delta Z_i$.

(d) If $\beta_j, \beta_k \geq 0$ or $\beta_j, \beta_k \leq 0$, then $\beta_j - \beta_k$ has the same sign as $\Delta Y_j/\Delta Z_i - \Delta Y_k/\Delta Z_i$.

We will draw on the results presented in Proposition 5 throughout our subsequent analysis.

Testing for Social Multipliers

We begin by showing how to use OLS estimands to draw inference about individual-specific social multipliers $M_{(i)}^{\text{heterog.}}$ and aggregate social multipliers $M^{\text{heterog.}}$, which are both defined in Table 1. If these multipliers are greater (less) than one, then it would suggest that spillover effects amplify (suppress) the impact of individual shocks on the average outcome in a group.

To learn about the social multipliers, we analyze the equilibrium effects $Y_{-i}/\Delta Z_i$, which represent spillover effects of Z_i on agent i's peers. By Proposition 5, we can assess whether social multipliers are greater (less) than one by evaluating the signs of these reduced form quantities. Although we are unable to compute $\{\Delta \bar{Y}_{-i}/\Delta Z_i\}_{i=1}^{J}$ within every group, we can estimate the average reduced form effects $\{\text{E}(\Delta \bar{Y}_{-i}/\Delta Z_i)\}_{i=1}^{J}$ using OLS regressions. With these estimates, we can test whether social multipliers are greater than or less than one for a subset of groups in the population, providing insight into the role of network spillovers. For example, a rejection of the null $H_0 : \text{E}(\Delta \bar{Y}_{-i}/\Delta Z_i) \leq 0$ implies that $\text{P}(M_{(i)}^{\text{heterog.}} > 1) > 0$.

Testing for Positive Interaction Effects

Next, we show how to test for positive (or negative) interaction effects among agents in the population. Recall that positive interaction effects indicate strategic complementarity, which is consistent with classical peer effects, but is inconsistent with household labor supply and oligopoly. In contrast, negative interaction effects indicate strategic substitutability, which is consistent with household labor supply and oligopoly, but not with classical peer effects.

By Proposition 5, the sign of the interaction effect β_j can be inferred from the individual spillover effect $\Delta Y_j/\Delta Z_i$, provided the sign of γ_i is known. Specifically, for two agents i and j where $\gamma_i > 0$, the interaction effect β_j of agent i always shares the same sign as $\Delta Y_j/\Delta Z_i$.

By this property, we can construct a test for the existence of positive interaction effects from OLS regressions. In particular, if we assume that $\mathrm{P}(\gamma_j \geq 0) = 1$, then we can assess whether $\beta_i > 0$ with positive probability by testing the null hypothesis $H_0 : \mathrm{E}(\Delta Y_i/\Delta Z_j) \leq 0$.

In some cases, it may not be feasible to regress the outcomes Y_i on the entire vector Z. Moreover, if the interaction effects are heterogeneous within groups, then using an alternative regression based on averages of $\{Z_j\}_j$ introduces omitted variable bias. This bias confounds our ability to recover the average individual spillover effects $\mathrm{E}(\Delta Y_i/\Delta Z_j)$, which prevents us from conducting the tests outlined above. Fortunately, we can still test for the presence of endogenous interaction effects even when running a correctly specified regression is infeasible.

Lemma 7. Define $\beta^{\mathrm{OLS}}_{Y_i,\bar{Z}_{-i}}$ to be the coefficient on $\bar{Z}_{-i}$ in an OLS regression of Y_i on $(1, Z_i, \bar{Z}_{-i})$. If this estimand is nonzero, then the interaction effect β_i is nonzero with positive probability.

Lemma 7 provides a way to test for endogenous interaction effects, even in the presence of heterogeneous effects, using an OLS regression of Y_i on $(1, Z_i, \bar{Z}_{-i})$. However, it is important to note that this regression does not allow us to determine the sign of the interaction effects.

Testing for the Relative Strengths of Interaction Effects

We can also use OLS to test for the relative strengths of interaction effects. Specifically, for two distinct agents j and k in the group, we may want to empirically assess whether $\beta_j \geq \beta_k$. For example, do female or male students face more social pressure? Do husbands or wives exhibit higher second earner effects? What types of firms have larger conduct parameters?

To conduct this test, we draw on Proposition 5. If β_j and β_k share the same sign and if $\gamma_i > 0$, then difference between agents' interaction effects, $\beta_j - \beta_k$, always has the same sign as the difference in individual spillover effects, $\Delta Y_j/\Delta Z_i - \Delta Y_k/\Delta Z_i$ for any third agent $i \notin \{j, k\}$. Under a monotonicity assumption, $\mathrm{P}(\gamma_i \geq 0) = 1$, we can assess whether $\beta_i > \beta_j$ with positive probability by testing the null hypothesis $H_0 : \mathrm{E}(\Delta Y_i/\Delta Z_k) \leq \mathrm{E}(\Delta Y_j/\Delta Z_k)$.

Testing for Bounded Spillovers

Using OLS regressions, we can also test Assumption III, which states that $\beta_i \in (1-N, 1)$ for every agent i. One consequence of this assumption is that $\Delta\bar{Y}/\Delta Z_i$ has the same sign as γ_i. Using this property, we can test $\mathrm{P}(1-N < \beta_i < 1) = 1$ through the null $H_0 : \mathrm{E}(\Delta\bar{Y}/\Delta Z_i) > 0$ as long as we maintain a monotonicity assumption that $\mathrm{P}(\gamma_i \geq 0) = 1$. Rejecting this test means that the spillovers are unbounded, which suggests that the model is likely misspecified.

4 Empirical Applications

We now examine two applications: peer effects in Kenyan primary schools (Duflo et al., 2011) and strategic pricing decisions of cocoa traders in Sierra Leone (Casaburi & Reed, 2022). Both studies adopt a linear-in-means model with constant interaction effects. In each case, the model is over-identified, as individual-level shifters affect the outcomes of multiple agents in a group. We exploit this over-identification to test for constant interaction effects, finding that these tests are rejected in both applications.[22] We then reanalyze the estimates under heterogeneous effects, drawing insights about endogenous interactions and social multipliers.

4.1 Classroom Peer Effects in Kenya

Our first application comes from Duflo et al. (2011), who study peer effects and the impact of ability tracking in primary schools in Kenya. The study includes 121 schools, each assigning students to one of two classrooms. Students in *treatment* schools are assigned to classrooms based on ability, as measured by their baseline test score, while students in *control* schools are randomly assigned. Following Duflo et al. (2011), we restrict the sample to the control group. This sample is composed of 2,849 students over 61 schools, each split into two rooms.[23]

To measure peer effects in classrooms, Duflo et al. (2011) consider the following model:

$$Y_i = \beta\bar{Y}_{-i} + Z_i'\gamma + \nu_s + \varepsilon_i, \tag{23}$$

where Y_i is the endline test score of a student i, $\bar{Y}_{-i}$ is the average endline test score of i's classmates, Z_i is a vector of controls that includes i's own baseline score, and ν_s is a school fixed effect. The authors use the average baseline score of i's classmates $\bar{Z}_{-i}$ as an instrument for $\bar{Y}_{-i}$. As outcome variables, they consider math, reading, and total endline test scores.[24]

4.1.1 OLS/IV Estimates and Over-identification Tests

Table 3 presents results from our implementation of linear peer effects estimators. The first three columns of Panel A give OLS estimates from regressing Y_i on Z_i and $\bar{Z}_{-i}$ with school fixed effects, the same specification used in Duflo et al. (2011). In the classical linear-in-means

[22]As usual, we need to maintain the assumption of instrument exogeneity for the over-identifying restrictions to be a test of homogeneous interaction effects. Otherwise, it is a test of multiple model assumptions.

[23]After removing missing data, we retain 2,190 students over 48 schools.

[24]Equation (23) corresponds to (E4) in the original paper, with the notation adjusted to align with ours.

model with equal class sizes, these OLS regressions recover two key economic quantities: the spillover effect, $\Delta Y_i/\Delta Z_j$, $j \neq i$, of peer j's baseline score on student i's endline score and the total individual effect, $\Delta Y_i/\Delta Z_i$, of student i's own baseline score on her endline score.

The last three columns of Panel A give estimates from OLS regressions of $\bar{Y}_{-i}$ on Z_i and $\bar{Z}_{-i}$, again with school fixed effects. Under Assumptions C.1 and C.4, Lemma 1 tells us that the coefficient on Z_i in these regressions should equal the coefficient on $\bar{Z}_{-i}$ in the first set of regressions. However, we find strong evidence that these coefficients differ, suggesting that Assumption C.1 and/or C.4 is violated and that the peer effects may differ within classrooms.

Table 3: Classroom Peer Effects—Primary Schools in Kenya

	Own Endline Score			Peers' Mean Endline Score		
	Total (1)	Math (2)	Literature (3)	Total (4)	Math (5)	Literature (6)
Panel A. Reduced Form						
Own Baseline Score	0.507***	0.496***	0.413***	0.007**	0.006*	0.007**
	(0.026)	(0.022)	(0.030)	(0.003)	(0.003)	(0.003)
Peers' Mean Baseline Score	0.345**	0.324**	0.291**	0.788***	0.697***	0.704***
	(0.150)	(0.160)	(0.131)	(0.157)	(0.174)	(0.134)
Observations	2,188	2,188	2,189	2,188	2,188	2,189
	One Instrument Spec.			Multiple Instrument Spec.		
	Total	Math	Literature	Total	Math	Literature
Panel B. Instrumental Variables						
Peers' Mean Endline Score	0.444***	0.469***	0.422***	0.424***	0.488***	0.487***
	(0.117)	(0.124)	(0.120)	(0.094)	(0.103)	(0.117)
First-Stage F-Stat	371.8	371.6	1970	293.4	463.4	590.9
Sargan-Hansen Test[a]				15.12	12.53	12.76
				(0.004)	(0.014)	(0.013)
Observations	2,188	2,188	2,189	2,188	2,188	2,188

Notes. Data comes from Duflo et al. (2011). Following the authors' specifications, we include school fixed effects and controls for gender, age, and being assigned to the contract teacher. Columns (1)-(3) in Panel B use peers' mean baseline score as an excluded instrument. Columns (4)-(6) in Panel B use as excluded instruments: peers' mean baseline score, peers' minimum and maximum baseline scores, and mean baseline scores of male and female peers. Standard errors clustered at the school level.

[a]We report the Sargan-Hansen χ^2_4 test statistic with the corresponding p-value in parentheses below. *p<0.1; **p<0.05; ***p<0.01.

The first three columns of Panel B report estimates for the main IV specification. For a classical linear-in-means model, these regressions would recover the constant peer effect β. The last three columns of Table 3, Panel B, report estimates from alternate IV specifications that use multiple excluded instruments. In addition to $\bar{Z}_{-i}$, we include four more instrumental variables: *(1)* minimum baseline score of peers, *(2)* maximum baseline score of peers, *(3)* average baseline score among female peers, and *(4)* average baseline score among male peers.

If the peer effects are constant across classrooms, then by Lemma 2, any combination of instruments should yield the same IV estimand. However, if the peer effects vary, then the IV estimand will depend on the particular choice of instruments used. To test for constant effects in the model, we conduct a Sargan–Hansen test for over-identifying restrictions using all five excluded instruments. This test allows us to determine the validity of over-identifying restrictions using any linear combination of the excluded instruments. We find that this test is rejected at the significance level 0.05, implying that peer effects differ between classrooms.

4.1.2 Reanalysis under Heterogeneous Interaction Effects

Motivated by these findings, we re-analyze the estimates in Table 3 under the linear-in-means model with heterogeneous interaction effects. Our analysis leverages two observations about the empirical setting. First, Assumption PM is likely to hold because students' baseline test scores are expected to have a nonnegative impact on their endline scores, making it plausible that $\mathrm{P}(\gamma_i \geq 0) = 1$ for all i. Second, Assumption NNW is likely to hold, as the experimental design ensures that baseline scores $\{Z_j\}_{j=1}^N$ are uncorrelated after conditioning on the school.

Learning from OLS Estimates

Consider the OLS estimates reported in the first three columns of Table 3, Panel A. Under heterogeneous interaction effects, the coefficient on Z_i represents the average total individual effect $\mathrm{E}(\Delta Y_i/\Delta Z_i)$—that is, the effect of a student's baseline test score on his/her own endline test score, after accounting for spillovers. We estimate that, on average, scoring 1 point higher on the baseline test would lead a student to score about 0.5 points on the endline test.

In these OLS regressions, the coefficient on peers' mean baseline score $\bar{Z}_{-i}$ is estimated to be positive and statistically significant. By Lemma 7, this result allows us to infer that peer effects are present in at least some classrooms. Nevertheless, under heterogeneous effects, we cannot use this estimate to infer the sign of these peer effects, even though they do exist.[25]

Learning from IV Estimates

Consider the IV estimates in the first three columns of Table 3, Panel B, where we estimate positive and statistically significant IV estimands of approximately 0.45. Under heterogeneous effects, these estimands represent weighted averages of peer effects β_i across students.

Given that Assumptions PM and NNW hold, Proposition 3 implies that the IV estimand

[25] In this application, it is infeasible to regress the outcomes Y on the entire vector $\tilde{Z} = (1, Z')'$ as it requires labeling each student i in a way that is consistent across classrooms. This task may be straightforward in certain applications, e.g., when studying labor supply in two-person households where there is always one primary earner. However, it is impractical in other cases where the number and composition of agents in a group varies. Also, when N is large, there could be more parameters to estimate than there are observations.

is a causal parameter, representing a positively weighted average of peer effects.[26] Moreover, if we further assume that all peer effects are nonnegative, such that $\mathrm{P}(\beta_i \geq 0) = 1$ for all i, then Proposition 4 implies that the IV estimands serve as upper bounds on the average peer effect $\mathrm{E}(\beta_i)$. We thus conclude that a 1 point increase in peers' average test scores would not raise a student's own score by more than about 0.45 points on average. This upper bound is high, which suggests that peer effects could have a substantial impact on student outcomes.

Testing for Social Multipliers

In the last three columns of Table 3, Panel A, we estimate a statistically significant, positive regression coefficient on a student's own baseline test score Z_i, corresponding to the average equilibrium effect $\mathrm{E}(\Delta\bar{Y}_{-i}/\Delta Z_i)$.[27] By Proposition 5, this result tells us that social multipliers must exceed one in at least some classrooms. In such settings, factors that boost one student's achievement are amplified through social interactions, raising overall performance.

4.2 Strategic Pricing Decisions in Sierra Leone

Our second application builds on the analysis conducted by Casaburi & Reed (2022), who study the strategic behavior of traders purchasing cocoa from farmers in Sierra Leone. During an experiment conducted from October to December 2011, half of the 80 traders in the sample were randomly assigned a subsidy of 150 leones per pound of cocoa sold at village markets. Data on prices and quantities from these transactions was subsequently collected for analysis.

Casaburi & Reed (2022) specify a model of imperfect competition among buyers. Each market consists of N buyers and a unit measure of homogenous producers. The price P_i that a buyer i pays to producers is given by the inverse supply $P_i = \lambda + \kappa Q_i + \theta \sum_{j \neq i} Q_j$, which is micro-founded by assuming there exists a representative producer with a love for variety.[28] A buyer's profit function equals $\Pi_i = Q_i(v + sZ_i - P_i)$, where v denotes the wholesale price net of costs and Z_i indicates whether the buyer is randomly assigned a subsidy valued at s.

In equilibrium, the buyers choose their quantities Q_i to maximize profit, while accounting for optimal decisions $\{Q_j\}_{j \neq i}$ of their competitors. The profit-maximizing quantities satisfy

[26] Specifically, this IV estimand places larger weights on students for which $\bar{Y}_{-i}$ is more responsive to $\bar{Z}_{-i}$.

[27] This interpretation assumes that students' baseline scores $\{Z_j\}_{j=1}^{N}$ are uncorrelated with one another, which is implied by the experimental design, as students are randomly assigned to classrooms within a school.

[28] Following footnote 6 in Casaburi & Reed (2022), a producer's profit is: $V(P, Q) = Q_0 + \sum_{i=1}^{N} P_i Q_i - C(Q)$, where $C(Q) = \lambda \sum_{i=1}^{N} Q_i + \frac{1}{2}\kappa \sum_{i=1}^{N} Q_i^2 + \theta \sum_{j \neq i} Q_i Q_j$ is the cost of production, and Q_0 is any unsold output.

the linear-in-means model with constant effects, where the interaction effect β is $\theta(N-1)/2\kappa$.

$$\begin{aligned} Q_i &= \frac{v-\lambda}{2\kappa} - \frac{\theta}{2\kappa}\sum_{j\neq i} Q_j + \frac{s}{2\kappa} Z_i \\ &= \underbrace{\alpha}_{(v-\lambda)/2\kappa} + \underbrace{\frac{\beta}{N-1}}_{-\theta/2\kappa} \sum_{j\neq i} Q_j + \underbrace{\gamma}_{s/2\kappa} Z_i, \quad \text{for } i \in \{1,\dots,N\}. \end{aligned} \tag{24}$$

In this setting, we can interpret $\theta/2\kappa$ as a conduct parameter that measures how a buyer i's demand depends on the total quantity purchased by i's competitors. Under constant effects, the conduct parameter is identified from IV, where the quantity purchased by i's competitors $\sum_{j\neq i} Q_j$ is instrumented by the treatment statuses of i's competitors, denoted by $\{Z_j\}_{j\neq i}$.[29]

4.2.1 OLS/IV Estimates and Over-identification Tests

Table 4 presents our implementation of linear peer effects estimators. The first two columns of Panel A report OLS estimates from regressing a buyer i's purchases Q_i on her own treatment status Z_i and the number of treated competitors $\sum_{j\neq i} Z_j$, with and without trader controls. Under constant effects, this regression recovers the spillover effect of a competitor j's subsidy on a trader i's purchases, as well as the total individual effect of a trader i's subsidy on her own purchases. The last two columns of Panel A report estimates from regressing $\sum_{j\neq i} Q_j$ on Z_i and $\sum_{j\neq i} Z_j$, with and without trader controls. If Assumptions C.1 and C.4 hold, then by Lemma 1, the coefficient on $\bar{Z}_{-i}$ in a regression of Y_i on Z_i and $\bar{Z}_{-i}$ should match the coefficient on Z_i in a regression of $\bar{Y}_{-i}$ on Z_i and $\bar{Z}_{-i}$. We are unable to reject this in the data.

The first two columns of Panel B present estimates from IV regressions of Q_i on $\sum_{j\neq i} Q_j$, where the number of treated competitors $\sum_{j\neq i} Z_j$ is the excluded instrument. In the classical linear-in-means model, this regression recovers the conduct parameter $-\theta/2\kappa$. The last two columns of Table 4, Panel B, report estimates from alternate IV specifications using multiple instruments. In addition to $\sum_{j\neq i} Z_j$, we introduce three extra instruments: *(1)* number of treated competitors who have access to a storage facility, *(2)* number of treated competitors older than the median age (37), and *(3)* number of treated competitors with baseline sales above the median (300 lbs of cocoa). Each of these instruments is valid by the same identification arguments used in the original paper. We use these over-identified regressions to test whether all traders share a common conduct parameter. We then conduct a Sargan–Hansen test for over-identifying restrictions using all four excluded instruments. From this exercise,

[29] Casaburi & Reed (2022) do not run this IV regression since they never explicitly define a market in their empirical analysis. Rather, they rely on additional model assumptions to estimate the market size N while never explicitly assigning traders to markets. To conduct our analysis, however, we need to know which traders belong to which markets. We achieve this objective by defining a market as the interaction between a week and a chiefdom, which represents a small administrative unit in Sierra Leone. In the data, we find that 90% of traders operate in a single chiefdom in a given week and that over 98% of traders make more than half of their sales in the same chiefdom. We leverage this observation to assign traders to chiefdoms.

Table 4: Strategic Interactions—Cocoa Traders in Sierra Leone

	Trader Quantity		Competitors' Total Quantity	
	(1)	(2)	(1)	(2)
Panel A. Reduced Form				
Treatment Trader	416.663***	454.895***	-166.995	-61.516
	(45.733)	(49.594)	(248.156)	(267.626)
Number of Treated Competitors	-10.733***	-7.423**	507.685***	522.394***
	(2.975)	(3.697)	(16.141)	(19.948)
Observations	610	602	610	602
Trader Controls		X		X
	One Instrument Spec.		**Multiple Instrument Spec.**	
	(1)	(2)	(1)	(2)
Panel B. Instrumental Variables				
Competitors' Total Quantity	-0.007	-0.020***	-0.004	-0.018***
	(0.006)	(0.007)	(0.006)	(0.007)
First-Stage F-Stat	23.06	14.15	22.90	14.09
Sargan-Hansen Test[a]			9.82	12.35
			(0.02)	(0.006)
Observations	610	602	610	602
Trader Controls		X		X

Notes. Data comes from Casaburi & Reed (2022). Following the original paper, we include week fixed effects. Trader controls are: baseline pounds of cocoa sold, number of villages where trader operates, baseline share of suppliers receiving credit from trader, age, years working with wholesaler, ownership of a cement or tile floor, mobile phone, and access to a storage facility. Sample sizes differ between (1) and (2) due to missing data about trader controls.

[a]We report a Sargan-Hansen χ^2_3 test statistic with a corresponding p-value in parentheses. *p<0.1; **p<0.05; ***p<0.01.

we find strong evidence against the constant effects assumption. This suggests that different traders likely respond strategically in different ways to their competitors' pricing decisions.

4.2.2 Reanalysis under Heterogeneous Interaction Effects

Motivated by these findings, we reanalyze the estimates in Table 4 under the linear-in-means model with heterogeneous effects. To better interpret our findings, we make two observations. First, Assumption PM is likely to hold since the coefficient γ_i is proportional to the subsidy s, which is homogeneous within and across markets. Second, Assumption NNW is likely to hold, as the experimental design ensures that treatments $\{Z_j\}_{j=1}^N$ are mutually uncorrelated.

Learning from OLS Estimates

Consider the OLS estimates in the first two columns of Table 4, Panel A. Under heterogeneous effects, the coefficient on Z_i recovers $\mathrm{E}(\Delta Q_i/\Delta Z_i)$, which represents the average effect of receiving a subsidy on a trader's own purchases, after accounting for spillovers. We estimate that, on average, the subsidy leads traders to buy about 400 more pounds cocoa from farmers.

In this OLS regression, the coefficient on the number of treated competitors $\sum_{j\neq i} Z_j$ is estimated to be negative and statistically significant. By Lemma 7, this finding suggests that the conduct parameters $\theta_i/2\kappa_i$ are nonzero with positive probability. Therefore, at least some traders exhibit strategic interactions, which tells us that markets are imperfectly competitive.

Learning from IV Estimates

Consider the IV estimates in the first two columns in Panel B. After including trader controls, we estimate a significant, negative IV estimand of -0.02. Under heterogeneous interaction effects, this estimand corresponds to a weighted average of conduct parameters among traders.

Since Assumptions PM and NNW are plausible in this environment, we conclude from Proposition 3 that the IV estimand is a causal parameter, representing a positively-weighted average of conduct parameters.[30] Moreover, as the conduct parameters $\theta_i/2\kappa_i$ are positive by construction, the IV estimand gives an upper bound on the average conduct parameter $\mathrm{E}(\theta_i/2\kappa_i)$ among traders. We conclude that, on average, raising a competitors' cocoa purchases by 1 pound does not reduce a trader's own purchases by more than 0.02 pounds. This upper bound is low, which suggests that strategic interactions are limited in this context.

Testing for Social Multipliers

In this application, we find no evidence of social multiplier effects. To see why, consider the OLS estimates in the last two columns in Table 4, Panel A. The coefficient on Z_i corresponds to $\mathrm{E}\left(\Delta\big(\sum_{j\neq i} Q_j\big)/\Delta Z_i\right)$, which measures the average effect of one trader i's treatment status on the total quantity of his/her competitors, after accounting for spillovers.[31] We estimate this coefficient to be small and statistically insignificant, indicating that there is no social multiplier in this setting. We therefore conclude that the strategic interactions have little to no material impact on how changes in traders' demand or costs affect overall market output.

5 Conclusion

We analyzed a general class of linear simultaneous equations models where agents are influenced by the average outcome of their peers. Our framework nests the classical linear-in-means model (Manski, 1993). Moreover, we extended the model to allow for both positive

[30] Larger weights placed on traders whose competitors' purchases are more responsive to receiving subsidies.

[31] As in the first application, this interpretation requires that $\{Z_j\}_{j=1}^N$ are uncorrelated with one another. This condition is ensured by the experimental protocols, as a trader's treatment status is randomly assigned.

and negative interaction effects that differ within and across groups. We showed that the assumption of uniform interaction effects significantly limits the scope of economic behavior, making the model unsuitable for many real-world applications. By allowing for heterogeneous effects, we demonstrated that the model can be applied more broadly to study a wide range of network settings, such as joint labor supply decisions within households and strategic interactions between firms. Using the heterogeneous effects framework, we examined what insights are gained from linear peer effects estimators. We found that linear OLS and IV regressions can be used to draw informative inferences about endogenous interaction effects and social multipliers, even while these methods do not yield point identification. We applied our results to two applications from Duflo et al. (2011) and Casaburi & Reed (2022).

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Appendix

Proof of Proposition 1

Consider a group with N agents, where agents' outcomes are defined by the system (1).[32] To prove Proposition 1, we begin by defining the reduced form system using matrix notation.

$$Y = \det(I - \mathbf{B})^{-1}\mathbf{A}[\alpha + \text{diag}(\gamma)Z],$$

where $\mathbf{A} = \text{adj}(I - \mathbf{B})$ is the adjugate of $I - \mathbf{B}$, and $\det(I - \mathbf{B})$ is the determinant of $I - \mathbf{B}$. By definition, $\mathbf{A}$ is equal to the transpose of the matrix of cofactors of $I - \mathbf{B}$. In particular, the individual entries $\{A_{ij}\}_{i,j}$ of the matrix $\mathbf{A}$ are defined so that:

$$A_{ij} = (-1)^{i+j} \times \det([I - \mathbf{B}]_{-j,-i}),$$

where $[I - \mathbf{B}]_{-j,-i}$ is a submatrix formed by removing the jth row and ith column of $I - \mathbf{B}$.

We want to derive alternate expressions for $\{A_{ij}\}_{i,j}$ that are not in matrix form. To do so, we write $A_{ij} = (-1)^{i+j} \times \det\big(\mathbf{C}(i,j) - (N-1)^{-1}\beta_{-j}\mathbf{1}'_{(N-1)\times 1}\big)$, where $\mathbf{C}(i,j) \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix that is given by $\mathbf{C}(i,j) = I_{-j,-i}(\mathbf{1}_{(N-1)\times 1} + (N-1)^{-1}\beta_{-j})$. This matrix satisfies:

$$\det\big(\mathbf{C}(i,j)\big) = \mathbb{1}\{i = j\} \times \prod_{\ell \neq j}\Big(1 + \frac{\beta_\ell}{N-1}\Big)$$

$$\text{adj}\big(\mathbf{C}(i,j)\big) = \begin{cases} \text{diag}\left(\left\{\prod\limits_{\ell \notin \{k,j\}}\Big(1 + \dfrac{\beta_\ell}{N-1}\Big)\right\}_{k \neq j}\right) & \text{if } i = j \\ \left[(-1)^{i+j-1} \times \prod\limits_{\ell \notin \{i,j\}}\Big(1 + \dfrac{\beta_\ell}{N-1}\Big)\right][\mathrm{e}_j]_{-i}[\mathrm{e}_i]'_{-j} & \text{if } i \neq j \end{cases}$$

Then, by the matrix determinant lemma, the diagonal entries $\{A_{jj}\}_{j=1}^N$ of $\mathbf{A}$ are equal to:

$$\begin{aligned} A_{jj} &= \det\big(\mathbf{C}(j,j)\big) - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\,\text{adj}\big(\mathbf{C}(j,j)\big)\beta_{-j} \\ &= \prod_{\ell \neq j}\Big(1 + \frac{\beta_\ell}{N-1}\Big) - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\,\text{diag}\left(\left\{\prod_{\ell \notin \{k,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\right\}_{k \neq j}\right)\beta_{-j} \\ &= \prod_{\ell \neq j}\Big(1 + \frac{\beta_\ell}{N-1}\Big) - \sum_{k \neq j}\left[\frac{\beta_k}{N-1}\prod_{\ell \notin \{k,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\right] \end{aligned}$$

[32] To simplify the notation, we will omit group subscripts and treat Z_i as a one-dimensional variable.

Moreover, by the exact same reasoning, the off-diagonal entries $\{A_{ij}\}_{i \neq j}$ of $\mathbf{A}$ are equal to:

$$\begin{aligned}
A_{ij} &= (-1)^{i+j} \times \Big[\det\big(\mathbf{C}(i,j)\big) - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\operatorname{adj}\big(\mathbf{C}(i,j)\big)\beta_{-j}\Big] \\
&= (-1)^{i+j} \times \Big[0 - \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1}\Big[(-1)^{i+j-1} \times \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big)\Big][\mathrm{e}_j]_{-i}[\mathrm{e}_i]'_{-j}\beta_{-j}\Big] \\
&= \frac{1}{N-1}\mathbf{1}'_{(N-1)\times 1} \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big)[\mathrm{e}_j]_{-i}[\mathrm{e}_i]'_{-j}\beta_{-j} \\
&= \frac{\beta_i}{N-1} \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big)
\end{aligned}$$

Now that we have derived these expressions for $\{A_{ij}\}_{i,j}$, our next step is to re-write the determinant of $I - \mathbf{B}$ so that it is not in matrix form. To do so, we take the following steps:

$$\begin{aligned}
\det(I - \mathbf{B}) &= \det\Big[I + \frac{1}{N-1}\mathrm{diag}(\beta) - \frac{1}{N-1}\beta\mathbf{1}'_{N\times 1}\Big] \\
&= \det\Big[I + \frac{1}{N-1}\mathrm{diag}(\beta)\Big]\Big(1 - \frac{1}{N-1}\mathbf{1}'_{N\times 1}\Big[I + \frac{1}{N-1}\,\mathrm{diag}(\beta)\Big]^{-1}\beta\Big)
\end{aligned}$$

For any agent $i \in \{1, \ldots, N\}$, this determinant can be reformulated as:

$$\begin{aligned}
\det(I - \mathbf{B}) &= \prod_{\ell=1}^{N}\Big(1 + \frac{\beta_\ell}{N-1}\Big) \times \Big[1 - \sum_{j=1}^{N}\frac{\beta_j}{N-1}\Big(1 + \frac{\beta_j}{N-1}\Big)^{-1}\Big] \\
&= \prod_{\ell \neq i}\Big(1 + \frac{\beta_\ell}{N-1}\Big) \times \Big[1 - \Big(1 + \frac{\beta_i}{N-1}\Big)\sum_{j \neq i}\frac{\beta_j}{N-1}\Big(1 + \frac{\beta_j}{N-1}\Big)^{-1}\Big] \\
&= A_{ii} - \frac{\beta_i}{N-1}\sum_{j \neq i}\Big[\frac{\beta_j}{N-1}\prod_{\ell \notin \{i,j\}}\Big(1 + \frac{\beta_\ell}{N-1}\Big)\Big]
\end{aligned}$$

By plugging in our expressions for $\{A_{ij}\}_{i,j}$ and $\det(I - \mathbf{B})$, we are now able to write down the ith reduced form equation for any agent $i \in \{1, \ldots, N\}$. This equation is given by:

$$\begin{aligned}
Y_i &= \frac{1}{\det(I - \mathbf{B})}\Big[A_{ii}(\alpha_i + \gamma_i Z_i) + \sum_{j \neq i} A_{ij}(\alpha_j + \gamma_j Z_j)\Big] \\
&= \alpha_i + \gamma_i Z_i + \frac{\sum_{j \neq i}\zeta_{ij} \times \Big[\frac{\beta_j}{N-1}(\alpha_i + \gamma_i Z_i) + (\alpha_j + \gamma_j Z_j)\Big]}{\det(I - \mathbf{B})}
\end{aligned}$$

where $\zeta_{ij} = \frac{\beta_i}{N-1}\prod_{\ell \notin \{i,j\}}\big(1 + \frac{\beta_\ell}{N-1}\big)$. Next, for any $i \in \{1, \ldots, N\}$, consider the average outcome $\bar{Y}_{-i}$ among everyone excluding agent i. To derive an expression for $\bar{Y}_{-i}$, we write:

$$\begin{aligned}
\bar{Y}_{-i} &= \frac{1}{(N-1) \times \det(I - \mathbf{B})} \times (\mathbf{1}_{N\times 1} - \mathrm{e}_i)'\mathbf{A}[\alpha + \mathrm{diag}(\gamma)Z] \\
&= \frac{1}{(N-1) \times \det(I - \mathbf{B})} \times \sum_{j=1}^{N}\underbrace{\Big[\sum_{k \neq i} A_{kj}\Big]}_{c_{ij}}(\alpha_j + \gamma_j Z_j),
\end{aligned}$$

where the coefficient $c_{ii} = \sum_{k \neq i} A_{ki}$ is defined to be:

$$c_{ii} = \sum_{k \neq i} \left[\frac{\beta_k}{N-1} \prod_{\ell \notin \{k,i\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \right],$$

and where each of the coefficients $c_{ij} = \sum_{k \neq i} A_{kj}$, for $j \neq i$, is defined to be:

$$\begin{aligned}
c_{ij} &= A_{jj} + \sum_{k \notin \{i,j\}} A_{kj} \\
&= \prod_{\ell \neq j} \Big(1 + \frac{\beta_k}{N-1}\Big) - \sum_{k \neq j} \left[\frac{\beta_k}{N-1} \prod_{\ell \notin \{k,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \right] + \sum_{k \notin \{i,j\}} \left[\frac{\beta_k}{N-1} \prod_{\ell \notin \{k,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \right] \\
&= \prod_{\ell \neq j} \Big(1 + \frac{\beta_k}{N-1}\Big) - \frac{\beta_i}{N-1} \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \\
&= \Big(1 + \frac{\beta_i}{N-1} - \frac{\beta_i}{N-1}\Big) \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \\
&= \prod_{\ell \notin \{i,j\}} \Big(1 + \frac{\beta_\ell}{N-1}\Big)
\end{aligned}$$

After plugging in these expressions for $\{c_{ij}\}_{j=1}^N$, we arrive at the following equation:

$$\bar{Y}_{-i} = \frac{\sum_{j \neq i} c_{ij} \times \left[\frac{\beta_j}{N-1}(\alpha_i + \gamma_i Z_i) + (\alpha_j + \gamma_j Z_j) \right]}{(N-1) \times \det(I - \mathbf{B})}$$

By taking similar steps, we can derive an analogous expression for the the mean outcome $\bar{Y}$:

$$\bar{Y} = \frac{\sum_{j=1}^N c_j \times (\alpha_j + \gamma_j Z_j)}{N \times \det(I - \mathbf{B})}, \quad \text{where} \quad c_j = \prod_{\ell \neq j} \Big(1 + \frac{\beta_\ell}{N-1}\Big) \quad \text{for} \quad j \in \{1, \ldots, N\}$$

□

Necessary and Sufficient Conditions for a Unique Equilibrium

A unique equilibrium exists if and only if the determinant of $I - \mathbf{B}$ is nonzero. We write:

$$\begin{aligned}
\det(I - \mathbf{B}) &= \prod_{j=1}^N \Big(1 + \frac{\beta_j}{N-1}\Big) \times \left[1 - \sum_{i=1}^N \frac{\beta_i}{N-1} \Big(1 + \frac{\beta_i}{N-1}\Big)^{-1} \right] \\
&= \sum_{i=1}^N \left[\frac{1}{N} \prod_{j=1}^N \Big(1 + \frac{\beta_j}{N-1}\Big) - \frac{\beta_i}{N-1} \prod_{j \neq i} \Big(1 + \frac{\beta_j}{N-1}\Big) \right] \\
&= \Big(\frac{N-1}{N}\Big) \sum_{i=1}^N (1 - \beta_i) \prod_{j \neq i} (N - 1 + \beta_j)
\end{aligned}$$

So, for any $N \geq 2$, a unique equilibrium exists if and only if $\sum_{i=1}^N (1-\beta_i) \prod_{j \neq i} (N-1+\beta_j) \neq 0$.

Proof of Lemma 3

Let Z_j and Z_k be binary variables so that (Z_j, Z_k) takes values in $\{(0,0),(0,1),(1,0),(1,1)\}$. Given this set of feasible values, Assumption IAM consists of four separate restrictions:

(1) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \leq 0\right) = 1$

(2) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

(3) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} + \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} + \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

(4) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} - \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} - \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

As long as $\beta_i, \beta_j, \beta_k \in (-1, 1)$, the partial effects $\Delta \bar{Y}_{-i}/\Delta Z_j$ and $\Delta \bar{Y}_{-i}/\Delta Z_k$ have the same signs (respectively) as γ_j and γ_k. For this reason, restrictions *(1)* and *(2)* are equivalent to:

(1') $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$

(2') $\mathrm{P}(\gamma_k \geq 0) = 1$ or $\mathrm{P}(\gamma_k \leq 0) = 1$

When combined with *(1)* and *(2)*, the restrictions *(3)* and *(4)* can be reformulated as a single condition: either $\mathrm{P}(|\Delta \bar{Y}_{-i}/\Delta Z_j| \geq |\Delta \bar{Y}_{-i}/\Delta Z_k|) = 1$ or $\mathrm{P}(|\Delta \bar{Y}_{-i}/\Delta Z_j| \leq |\Delta \bar{Y}_{-i}/\Delta Z_k|) = 1$. We can re-interpret this condition as a statement about the random coefficients by writing:

(3') $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \geq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$ or $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} \leq \left|\frac{\gamma_j}{\gamma_k}\right|\right) = 1$

□

Proof of Lemma 4

Let Z_j and Z_k be continuous variables, and consider any two vectors (z_j, z_k) and (z_j', z_k') taken from the support of (Z_j, Z_k). The difference in $\bar{Y}_{-i}$ when evaluated at these vectors is:

$$\begin{aligned}\bar{Y}_{-i}(z_j, z_k) - \bar{Y}_{-i}(z_j', z_k') &= \left[\bar{Y}_{-i}(z_j, z_k) - \bar{Y}_{-i}(z_j', z_k)\right] + \left[\bar{Y}_{-i}(z_j', z_k) - \bar{Y}_{-i}(z_j', z_k')\right] \\ &= \frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \times (z_j - z_j') + \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \times (z_k - z_k')\end{aligned}$$

Assumption IAM requires that $\bar{Y}^{g_1}_{-i}(z_j, z_k) - \bar{Y}^{g_1}_{-i}(z_j', z_k')$ and $\bar{Y}^{g_2}_{-i}(z_j, z_k) - \bar{Y}^{g_2}_{-i}(z_j', z_k')$ share the same sign for any two groups g_1 and g_2. We show that the condition holds if and only if:

(1) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_j} \leq 0\right) = 1$

(2) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \geq 0\right) = 1$ or $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \leq 0\right) = 1$

(3) $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}/\Delta Z_j}{\Delta \bar{Y}_{-i}/\Delta Z_k} = a\right) = 1$ for some $a \in \mathbb{R}$

("$\Leftarrow$") Suppose Assumption IAM holds. Then, *(1)* and *(2)* apply for the same reason that they do in the binary case. To justify *(3)*, take (z_j, z_k) to be any vector that lies within the interior of the support of (Z_j, Z_k). Then, for groups g_1 and g_2, define the quantities:

$$z_j' = z_j - \left[\frac{\Delta \bar{Y}^{g_1}_{-i}}{\Delta Z_k} + \frac{\Delta \bar{Y}^{g_2}_{-i}}{\Delta Z_k}\right] \times \epsilon \quad \text{and} \quad z_k' = z_k + \left[\frac{\Delta \bar{Y}^{g_1}_{-i}}{\Delta Z_j} + \frac{\Delta \bar{Y}^{g_2}_{-i}}{\Delta Z_j}\right] \times \epsilon,$$

where $\epsilon > 0$ is chosen to be sufficiently small so that (z_j', z_k') lies inside the support of (Z_j, Z_k). In this case, the differences $\bar{Y}_{-i}^{g_1}(z_j, z_k) - \bar{Y}_{-i}^{g_1}(z_j', z_k')$ and $\bar{Y}_{-i}^{g_2}(z_j, z_k) - \bar{Y}_{-i}^{g_2}(z_j', z_k')$ are equal to:

$$\bar{Y}_{-i}^{g_1}(z_j, z_k) - \bar{Y}_{-i}^{g_1}(z_j', z_k') = \left(\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_j} \times \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_k}\right)\epsilon - \left(\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_k} \times \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_j}\right)\epsilon$$
$$\bar{Y}_{-i}^{g_2}(z_j, z_k) - \bar{Y}_{-i}^{g_2}(z_j', z_k') = \left(\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_k} \times \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_j}\right)\epsilon - \left(\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_j} \times \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_k}\right)\epsilon$$

Observe that the first equation is equal to the negative of the second equation. So, these differences can only share the same sign when they both equal zero. Specifically, we require:

$$\frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_j} \times \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_k} = \frac{\Delta \bar{Y}_{-i}^{g_1}}{\Delta Z_k} \times \frac{\Delta \bar{Y}_{-i}^{g_2}}{\Delta Z_j} \quad \Longleftrightarrow \quad \frac{\Delta \bar{Y}_{-i}^{g_1}/\Delta Z_j}{\Delta \bar{Y}_{-i}^{g_1}/\Delta Z_k} = \frac{\Delta \bar{Y}_{-i}^{g_2}/\Delta Z_j}{\Delta \bar{Y}_{-i}^{g_2}/\Delta Z_k}$$

This equation holds for any two groups g_1 and g_2. So, $\mathrm{P}\left(\frac{\Delta \bar{Y}_{-i}/\Delta Z_j}{\Delta \bar{Y}_{-i}/\Delta Z_k} = a\right) = 1$ for some $a \in \mathbb{R}$.

("$\Rightarrow$") Suppose that conditions *(1)*, *(2)*, and *(3)* apply. Then, for some constant $a \in \mathbb{R}$:

$$\bar{Y}_{-i}(z_j, z_k) - \bar{Y}_{-i}(z_j', z_k') = \frac{\Delta \bar{Y}_{-i}}{\Delta Z_k} \times \left[a \times (z_j - z_j') + (z_k - z_k')\right],$$

where $\Delta \bar{Y}_{-i}/\Delta Z_k$ retains the same sign across groups. Thus, Assumption IAM must apply. Note that we can re-write the conditions *(1)*, *(2)*, and *(3)* in terms of the random coefficients:

(1') $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$
(2') $\mathrm{P}(\gamma_k \geq 0) = 1$ or $\mathrm{P}(\gamma_k \leq 0) = 1$
(3') $\mathrm{P}\left(\frac{1+\frac{1}{2}\beta_j}{1+\frac{1}{2}\beta_k} = a \times \frac{\gamma_j}{\gamma_k}\right) = 1$ for some $a \in \mathbb{R}$

□

Proof of Lemma 5

For any $j \neq i$, consider any two vectors $(z_j, \{z_k\}_{k \notin \{i,j\}})$ and $(z_j', \{z_k\}_{k \notin \{i,j\}})$ in the support of Z_{-i}. By Lemma 1, the difference between the values of $\bar{Y}_{-i}$ evaluated at these vectors is:

$$\bar{Y}_{-i}(z_j, \{z_k\}_{k \notin \{i,j\}}) - \bar{Y}_{-i}(z_j', \{z_k\}_{k \notin \{i,j\}}) = \frac{\prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{N-1}\right) \times \gamma_j (z_j - z_j')}{(N-1) \times \det(I - \mathbf{B})}$$

As $\det(I - \mathbf{B}) > 0$ and $\prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{N-1}\right) > 0$ with probability 1, the PM condition requires:

$$\mathrm{P}\big(\gamma_j (z_j - z_j') \geq 0\big) = 1 \quad \text{or} \quad \mathrm{P}\big(\gamma_j (z_j - z_j') \leq 0\big) = 1,$$

which occurs if and only if $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$. This condition applies for all j.

□

Proof of Lemma 6

This result immediately follows from the observation that, under Assumption III, the spillover effect $\Delta \bar{Y}_{-i}/\Delta Z_j$ of Z_j on $\bar{Y}_{-i}$ always shares the same sign as γ_j for every $j \in \{1, \ldots, N\} \setminus i$.

□

Proof of Proposition 3

In this model, $\bar{Y}_{-i}$ is a linear function of Z. Therefore, we can write $\bar{Y}_{-i} = \pi_0 + \sum_{j=1}^{N} \pi_j Z_j$ for some parameters π_0 and $\{\pi_j\}_{j=1}^{N}$ that depend on the random coefficient vector $(\alpha, \beta, \gamma, \mathcal{N})$. Because the random coefficients are independent of Z, the conditional expectation of $\bar{Y}_{-i}$ given Z is equal to $\mathrm{E}(\bar{Y}_{-i}|Z) = \mathrm{E}(\pi_0) + \sum_{j=1}^{N} \mathrm{E}(\pi_j) Z_j$. Given these properties, we can write:

$$\begin{aligned}
\hat{\beta}_i^{\text{TSLS}}(z_i) \xrightarrow{p} \frac{\mathrm{Cov}(Y_i, \mathbf{L}(\bar{Y}_{-i}|Z_{-i})|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathbf{L}(\bar{Y}_{-i}|Z_{-i})|Z_i = z_i)} &= \frac{\mathrm{Cov}(Y_i, \mathrm{E}(\bar{Y}_{-i}|Z)|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathrm{E}(\bar{Y}_{-i}|Z)|Z_i = z_i)} \\
&= \sum_{j \neq i} \mathrm{E}(\pi_j) \times \frac{\mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, \mathrm{E}(\bar{Y}_{-i}|Z_{-i})|Z_i = z_i)} \\
&= \sum_{j \neq i} \mathrm{E}(\pi_j) \times \frac{\mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} \\
&= \sum_{j \neq i} \underbrace{\frac{\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)}{\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)}}_{\omega_j} \times \frac{\mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)}
\end{aligned}$$

By construction, the weights $\{\omega_j\}_{j \neq i}$ sum to one. In addition, we prove the following claim.

Claim 1. Suppose that Assumption NNW holds. Then ω_j will be non-negative for all $j \neq i$.

Proof. For $j \neq i$, the weight ω_j is non-negative if and only if its numerator and denominator have the same sign. So, $\{\omega_j\}_{j \neq i}$ are non-negative if and only if $\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ has the same sign as $\sum_{k \neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)$ for all $j \neq i$. Note that this statement is equivalent to the requirement that $\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ retains the same sign across all $j \neq i$. Therefore, for any $j, k \in \{1, \ldots, N\} \setminus i$, we rule out the case where:

$$\begin{aligned}
0 &> \mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i) \\
&= \mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)\,\mathrm{Cov}(Z_j, Z_k|Z_i = z_i) + \sum_{\ell \notin \{i,k\}} \mathrm{E}(\pi_j)\,\mathrm{E}(\pi_\ell)\,\mathrm{Cov}(Z_\ell, Z_j|Z_i = z_i) \\
0 &< \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i) \\
&= \mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)\,\mathrm{Cov}(Z_j, Z_k|Z_i = z_i) + \sum_{\ell \notin \{i,j\}} \mathrm{E}(\pi_k)\,\mathrm{E}(\pi_\ell)\,\mathrm{Cov}(Z_\ell, Z_k|Z_i = z_i)
\end{aligned}$$

These inequalities can be reformulated in terms of bounds on the covariance of Z_j and Z_k.[33]

$$-\sum_{\ell \notin \{i,j\}} \frac{\mathrm{E}(\pi_\ell)}{\mathrm{E}(\pi_j)} \mathrm{Cov}(Z_\ell, Z_k|Z_i = z_i) < \mathrm{Cov}(Z_j, Z_k|Z_i = z_i) < -\sum_{\ell \notin \{i,k\}} \frac{\mathrm{E}(\pi_\ell)}{\mathrm{E}(\pi_k)} \mathrm{Cov}(Z_\ell, Z_j|Z_i = z_i)$$

Therefore, the requirement that all the weights $\{\omega_j\}_{j \neq i}$ are non-negative is equivalent to the condition that $\mathrm{Cov}(Z_j, Z_k|Z_i = z_i)$ does not satisfy the inequalities above for any $j, k \neq i$.

□

[33] To see how, divide both inequalities by $\mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)$, which we assume is positive without loss of generality. If $\mathrm{E}(\pi_j)\,\mathrm{E}(\pi_k)$ is negative, then the inequalities flip, and the claim still holds as j and k are chosen arbitrarily.

Having proven this claim, the next step is to write down an expression for the TSLS estimand as a weighted average of individual β_i's. Consider the following decomposition:

$$
\begin{aligned}
\hat{\beta}_i^{\text{TSLS}}(z_i) \xrightarrow{p} \frac{\sum_{j\neq i} \mathrm{E}(\pi_j) \times \mathrm{Cov}(Y_i, Z_j|Z_i = z_i)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} &= \frac{\sum_{j\neq i} \mathrm{E}(\pi_j) \times \left(\sum_{\ell\neq i} \mathrm{E}(\beta_i\pi_\ell) \times \mathrm{Cov}(Z_\ell, Z_j|Z_i = z_i)\right)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} \\
&= \frac{\sum_{\ell\neq i} \mathrm{E}(\beta_i\pi_\ell) \times \left(\sum_{j\neq i} \mathrm{E}(\pi_j) \times \mathrm{Cov}(Z_\ell, Z_j|Z_i = z_i)\right)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} \\
&= \frac{\sum_{\ell\neq i} \mathrm{E}(\beta_i\pi_\ell) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_\ell|Z_i = z_i)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} \\
&= \mathrm{E}\left(\beta_i \times \frac{\sum_{\ell\neq i} \pi_\ell \times \mathrm{Cov}(\bar{Y}_{-i}, Z_\ell|Z_i = z_i)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)}\right)
\end{aligned}
$$

To obtain the second equation above, we switch the order of summation in the numerator. The final equation holds by linearity of expectation. In integral form, the TSLS estimand is:

$$
\beta_i^{\text{TSLS}}(z_i) = \int_{\text{supp}(\beta_i)} \beta_i \times \omega(\beta_i|z_i) d\beta_i,
$$

$$
\text{where:} \quad \omega(\beta_i|z_i) = \frac{\sum_{\ell\neq i} \mathrm{E}(\pi_\ell|\beta_i) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_\ell|Z_i = z_i)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k|Z_i = z_i)} f_{\beta_i}(\beta_i)
$$

The last step of this proof will be to demonstrate that the weights $\omega(\beta_i|z_i)$ are all non-negative as long as Assumptions PM and NNW are satisfied. We justify this claim below.

<u>*Claim 2.*</u> If Assumptions PM and NNW hold, then $\omega(\beta_i|z_i)$ is non-negative for every β_i.

Proof. Using Lemma 1, we can write the coefficient π_j, for any $j \neq i$, to be:

$$
\pi_j = \begin{cases} \gamma_j \times \dfrac{\prod_{\ell\notin\{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right)}{(|\mathcal{N}| - 1) \times \det(I - \mathbf{B})} & \text{if } j \in \mathcal{N} \\ 0 & \text{if } j \notin \mathcal{N} \end{cases}
$$

Here, $\prod_{\ell\notin\{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right) > 0$ and $\det(I - \mathbf{B}) > 0$ with probability one. Moreover, by PM, either $\gamma_j \geq 0$ with probability one or $\gamma_j \leq 0$ with probability one. Without loss of generality, assume that $\gamma_j \geq 0$ with probability one. Then $\mathrm{P}(\pi_j \geq 0) = 1$, which ensures that:

$$
\mathrm{E}(\pi_j) = \int_{-\infty}^{\infty} \pi_j f_{\pi_j}(\pi_j) d\pi_j = \int_0^{\infty} \pi_j f_{\pi_j}(\pi_j) d\pi_j \geq 0
$$

$$
\mathrm{E}(\pi_j|\beta_i) f_{\beta_i}(\beta_i) = \int_{-\infty}^{\infty} \pi_j f_{\pi_j|\beta_i}(\pi_j|\beta_i) f_{\beta_i}(\beta_i) d\pi_j = \int_0^{\infty} \pi_j f_{\pi_j,\beta_i}(\pi_j, \beta_i) d\pi_j \geq 0
$$

These inequalities imply that $\mathrm{E}(\pi_j)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ and $\mathrm{E}(\pi_j|\beta_i)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i) f_{\beta_i}(\beta_i)$ are either both non-negative or both non-positive across all $\beta_i \in \text{supp}(\beta_i)$. Moreover, as the index j was chosen arbitrarily, this relationship applies for all $j \in \{1, \ldots, N\} \setminus i$.

Assumption NNW ensures that $\mathrm{E}(\pi_j)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i)$ has the same sign across all $j \neq i$. Since these terms also share the same sign as $\mathrm{E}(\pi_j|\beta_i)\,\mathrm{Cov}(\bar{Y}_{-i}, Z_j|Z_i = z_i) f_{\beta_i}(\beta_i)$, for all $\beta_i \in \text{supp}(\beta_i)$ and $j \neq i$, we conclude that all the weights $\omega(\beta_i|z_i)$ would be non-negative. $\square$

Proof of Proposition 4

As a first step, we decompose the TSLS estimand to isolate the mean interaction effect.

$$\begin{aligned}
\beta_i^{\mathrm{TSLS}}(z_i) &= \frac{\sum_{j\neq i} \mathrm{E}(\beta_i \pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_i = z_i)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)} \\
&= \mathrm{E}(\beta_i) + \underbrace{\frac{\sum_{j\neq i} \mathrm{Cov}(\beta_i, \pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_i = z_i)}{\sum_{k\neq i} \mathrm{E}(\pi_k) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_k | Z_i = z_i)}}_{(*)}
\end{aligned}$$

Under Assumption NNW, the product $\mathrm{E}(\pi_j) \times \mathrm{Cov}(\bar{Y}_{-i}, Z_j | Z_i = z_i)$ has the same sign across all $j \neq i$. So, whenever $\mathrm{Cov}(\beta_i, \pi_j)$ has the same sign as $\mathrm{E}(\pi_j)$ for all $j \neq i$, the term $(*)$ will be positive. Alternatively, if $\mathrm{Cov}(\beta_i, \pi_j)$ and $\mathrm{E}(\pi_j)$ have opposite signs for all $j \neq i$, then the term $(*)$ will be negative. This reasoning leads us to the second step of the proof, where we show $\psi_i = \frac{1}{N-1} \sum_{j\neq i} \left[\beta_j \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{N-1}\right)\right]$ governs the sign of $\mathrm{Cov}(\beta_i, \pi_j)$ relative to $\mathrm{E}(\pi_j)$.

Pick any j, where $j \neq i$. By the PM condition, either $\mathrm{P}(\gamma_j \geq 0) = 1$ or $\mathrm{P}(\gamma_j \leq 0) = 1$. Without loss of generality, assume $\mathrm{P}(\gamma_j \geq 0) = 1$. Then, as shown in the proof of Theorem 2, the mean of π_j must be positive. Also, the Law of Total Covariance guarantees that:

$$\mathrm{Cov}(\beta_i, \pi_j) = \mathrm{E}\left(\mathrm{Cov}(\beta_i, \pi_j | \gamma_j, \beta_{-i}, \mathcal{N})\right) + \underbrace{\mathrm{Cov}\left(\mathrm{E}(\beta_i | \gamma_j, \beta_{-i}, \mathcal{N}), \mathrm{E}(\pi_j | \gamma_j, \beta_{-i}, \mathcal{N})\right)}_{=0}$$

Note that the second term on the right-hand-side is zero because $\mathrm{E}(\beta_i | \gamma_j, \beta_{-i}, \mathcal{N}) = \mathrm{E}(\beta_i)$. Following the proof of Lemma 1, the coefficient π_j can be expressed in terms of ψ_i by writing:

$$\begin{aligned}
\pi_j &= \mathbb{1}\{j \in \mathcal{N}\} \times \frac{\gamma_j \times \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right)}{(|\mathcal{N}| - 1) \times \det(I - \mathbf{B})} \\
&= \mathbb{1}\{j \in \mathcal{N}\} \times \frac{\gamma_j \times \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}|-1}\right)}{(|\mathcal{N}| - 1) \times \left[A_{ii} - \beta_i \times \psi_i / (|\mathcal{N}| - 1)^2\right]}
\end{aligned}$$

where A_{ii} depends only on β_{-i} and $\mathcal{N}$. To simplify notation, define the following parameters:

$$\begin{aligned}
\delta_{ij} &= \mathbb{1}\{j \in \mathcal{N}\} \times (|\mathcal{N}| - 1) \times \prod_{\ell \notin \{i,j\}} \left(1 + \frac{\beta_\ell}{|\mathcal{N}| - 1}\right) \\
\xi_i &= (|\mathcal{N}| - 1)^2 \times A_{ii}
\end{aligned}$$

These terms δ_{ij} and ξ_i depend only on β_{-i} and $\mathcal{N}$. Also, δ_{ij} is positive with probability one.

Using this new notation, we can write covariance between β_i and π_j to be:

$$
\begin{aligned}
\mathrm{Cov}(\beta_i, \pi_j) &= \mathrm{E}\left(\mathrm{Cov}\left(\beta_i, \frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} \Big| \gamma_j, \beta_{-i}, \mathcal{N} \right) \right) \\
&= \mathrm{E}\left(\mathrm{E}\left(\Big[\beta_i - \mathrm{E}(\beta_i)\Big] \times \Big[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} - \mathrm{E}\left(\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} \Big| \gamma_j, \beta_{-i}, \mathcal{N} \right) \Big] \Big| \gamma_j, \beta_{-i}, \mathcal{N} \right) \right) \\
&= \mathrm{E}\left(\mathrm{E}\left(\Big[\beta_i - \mathrm{E}(\beta_i)\Big] \times \Big[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} - \frac{\gamma_j \times \delta_{ij}}{\xi_i - \mathrm{E}(\beta_i) \times \psi_i} \Big] \Big| \gamma_j, \beta_{-i}, \mathcal{N} \right) \right) \\
&\quad + \mathrm{E}\left(\mathrm{E}\left(\Big[\beta_i - \mathrm{E}(\beta_i)\Big] \times \Big[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \mathrm{E}(\beta_i) \times \psi_i} - \mathrm{E}\left(\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} \Big| \gamma_j, \beta_{-i}, \mathcal{N} \right) \Big] \Big| \gamma_j, \beta_{-i}, \mathcal{N} \right) \right) \\
&= \mathrm{E}\left(\Big[\beta_i - \mathrm{E}(\beta_i)\Big] \times \Big[\frac{\gamma_j \times \delta_{ij}}{\xi_i - \beta_i \times \psi_i} - \frac{\gamma_j \times \delta_{ij}}{\xi_i - \mathrm{E}(\beta_i) \times \psi_i} \Big] \right) \\
&= \mathrm{E}\left(\psi_i \times \underbrace{\frac{\gamma_j \times \delta_{ij} \times \big[\beta_i - \mathrm{E}(\beta_i)\big]^2}{(\xi_i - \mathrm{E}(\beta_i) \times \psi_i)(\xi_i - \beta_i \times \psi_i)}}_{\geq 0 \text{ almost surely and } \neq 0 \text{ with positive probability}} \right)
\end{aligned}
$$

If $\psi_i > 0$ with probability one, then $\mathrm{Cov}(\beta_i, \pi_j) > 0$. Alternatively, if $\psi_i < 0$ with probability one, then $\mathrm{Cov}(\beta_i, \pi_j) < 0$. Therefore, we conclude that $\mathrm{Cov}(\beta_i, \pi_j)$ has the same (different) sign as $\mathrm{E}(\pi_j)$ whenever ψ_i is positive (negative) with probability one. Since j is chosen arbitrarily, this relationship holds for all $j \neq i$. By the arguments above, this property ensures that the term $(*)$ is positive if $\mathrm{P}(\psi_i > 0) = 1$ and that $(*)$ is negative if $\mathrm{P}(\psi_i < 0) = 1$. □

Derivation of Social Multipliers

We now derive a closed-form expression for the individual-specific social multiplier $M_{(i)}^{\text{heterog.}}$.

$$
\begin{aligned}
M_{(i)}^{\text{heterog.}} = \frac{\sum_{j=1}^{N} \Delta Y_j / \Delta Z_i}{\Delta Y_i / \Delta Z_i} &= \frac{\frac{\gamma_i \nu_i}{\det(I - \mathbf{B})}}{\gamma_i + \frac{\beta_i \gamma_i \left(\frac{1}{N-1} \sum_{j \neq i} \beta_j \nu_{ij} \right)}{(N-1)\det(I - \mathbf{B})}} \\
&= \frac{\left(1 + \frac{\beta_\ell}{N-1}\right)^{-1}}{1 - \sum_{j=1}^{N} \frac{\beta_j}{N-1}\left(1 + \frac{\beta_j}{N-1}\right)^{-1} + \frac{\beta_i}{N-1}\left(1 + \frac{\beta_i}{N-1}\right)^{-1} \left(\sum_{j \neq i} \frac{\beta_j}{N-1}\left(1 + \frac{\beta_j}{N-1}\right)^{-1} \right)} \\
&= \frac{\left(1 + \frac{\beta_\ell}{N-1}\right)^{-1}}{1 - \frac{\beta_i}{N-1}\left(1 + \frac{\beta_i}{N-1}\right)^{-1} + \left[\frac{\beta_i}{N-1}\left(1 + \frac{\beta_i}{N-1}\right)^{-1} - 1 \right] \left(\sum_{j \neq i} \frac{\beta_j}{N-1}\left(1 + \frac{\beta_j}{N-1}\right)^{-1} \right)} \\
&= \frac{\left(1 + \frac{\beta_i}{N-1}\right)^{-1}}{\left(1 + \frac{\beta_i}{N-1}\right)^{-1} \left[1 - \left(\sum_{j \neq i} \frac{\beta_j}{N-1}\left(1 + \frac{\beta_j}{N-1}\right)^{-1} \right) \right]} \\
&= \frac{1}{1 - \left(\sum_{j \neq i} \frac{\beta_j}{N-1}\left(1 + \frac{\beta_j}{N-1}\right)^{-1} \right)}
\end{aligned}
$$

Note that the derivation of the expression for the aggregate multiplier $M^{\text{heterog.}}$ is analogous.